Homework #10
Introduction to physical oceanography

1. **Gravity waves in a finite depth water:** Using the dispersion relation for gravity waves in a finite depth ocean (of depth H) which you found in the previous homework problem,

$$\omega^2(k) = gk\tanh(kH)$$

investigate the deep water (very large depth $H \gg k^{-1}$) and shallow water ($H \ll k^{-1}$) limits and show that these give the same results for the phase velocity which we found in class for deep water and shallow water waves. See Knauss for help on how to take these limits.

2. **Group and phase velocities:** (reproducing Knauss Figure 9.8)

   (a) Choose two slightly different wave numbers k and k' and calculate their frequency ω and ω' using the dispersion relation for deep gravity wave.

   (b) Plot in Matlab two waves of the form $cos(kx-\omega t)$ and $cos(k'x-\omega' t)$ at some time t_0 and the sum of the two waves (the three curves should be superimposed on the same graph).

   (c) Repeat the last item for three additional different times $t_0+\Delta t$, $t_0+2\Delta t$ and $t_0+3\Delta t$ where $\Delta t \approx 2\pi/\omega$.

   (d) All of the above four plots should be on the same page. Proceed by drawing lines passing through the nodes (zero points) of each of the two waves and of the envelope of their sum. Find the ratio between the speed of the envelope and that of the individual waves.

3. **Phase velocities in 2d:** you may use the Matlab program `phase_velocity2d.m` on the course home page for the followings.

   (a) Plot contours of the surface elevation for a deep gravity wave in two dimensions at $\eta(x,y,t) = \cos(kx+ly-\omega t)$ at a time t_0 in one color. Choose a case where $k = 5l$.

   (b) Superimpose on the previous plot another contour plot of the same wave at half the wave period later $t = t_0 + \frac{1}{2}2\pi/\omega$ using a different color.

   (c) Calculate from the plot the phase velocity in the x-direction, and in the y-direction.

4. **Phase and group velocities for internal Rossby and Kelvin waves:** the dispersion relation for internal Rossby waves was given in class as ($\omega = -\beta k/(k^2+l^2+L_R^{-2})$), where $L_R = \sqrt{g'H}/f$. for an internal Kelvin wave propagating in the x-direction, $\omega = \sqrt{g'H}k$. For both types of waves, do the followings:

   (a) Plot the frequency as function of k (both waves on the same figure); assume the wave length in the y direction to be 100 km.

   (b) Calculate the phase speed and group speed as function of k for a range of wave lengths from 10 to 1000 km. Plot both as function of k.

   (c) A storm passing over the Pacific ocean excites internal waves propagating in the east-west direction with a wavelength of 2000km. (i) What frequencies could these waves have (consider Poincare, Kelvin and Rossby waves)? (ii) What is the numerical value of the phase speed of all of these waves? (iii) The group speed? (iv) How long would the Rossby waves take to cross the Pacific ocean at latitude 30N? The Kelvin waves at the equator?

5. **Optional challenge problem: Group velocities in 2d:**

   (a) Plot contours of the surface elevation for a combination of two deep gravity wave trains in two dimensions

   $$\eta(x,y,t) = \cos(kx + ly - \omega t) + \cos((k+\delta)x + (l+\delta)y - (\omega + \delta\omega)t)$$

   at a time t_0 and at some later time. Choose a case where $k = 5l$ and $\delta = 0.2k$. Use the dispersion relation $\omega = \sqrt{gK}$, where $K = \sqrt{k^2 + l^2}$.

   (b) Calculate the group velocity in the direction of propagation of the wave by measuring the speed of the envelope propagation in your contour plot. Indicate on the plot the direction of the group velocity as an arrow. (Alternatively, you can plot the amplitude along the direction of propagation only, in which case you don't need to plot any contour plots.)

   (c) Plot the amplitude of the wave at $y = 0$ as function of x and at $x = 0$ as function of y.

   (d) Plot the same amplitudes of the previous section at a later time equal to 3 wave periods later $t = t_0 + 3(2\pi/\omega)$, super imposed on the previous plots, using a different color.

   (e) Calculate from the plot the group velocities in the x-direction, and in the y-direction.

   (f) Based on the results of the above questions, discuss the difference in the projection of the phase velocity and group velocity on the two axes. Explain why it makes sense that the group velocity is a vector while the phase velocity is not.