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Introduction to Physical Oceanography
Homework 7 - Solutions

1. Short review on coordinate systems: The unit vectors in Cartesian coordinates are: $\hat{x}$ and $\hat{y}$, while the unit vectors in polar coordinates are: $\hat{r}$ and $\hat{\theta}$. Some definitions:

$$\begin{aligned} x = &\ rcos(\theta); \quad r = &\sqrt{x^2+y^2} \\ y = &\ rsin(\theta); \quad \theta = &tan^{-1}\left(\tfrac{y}{x}\right) \end{aligned}$$

and

$$\begin{aligned} \hat{r} &= cos(\theta)\hat{x} + sin(\theta)\hat{y} = \frac{x\hat{x}+y\hat{y}}{\sqrt{x^2+y^2}} \\ \hat{\theta} &= -sin(\theta)\hat{x} + cos(\theta)\hat{y} = \frac{-y\hat{x}+x\hat{y}}{\sqrt{x^2+y^2}} \end{aligned}$$

where I used the identities

$$\begin{aligned} cos\left[tan^{-1}\left(\frac{y}{x}\right)\right] &= \frac{x}{\sqrt{x^2+y^2}} \\ sin\left[tan^{-1}\left(\frac{y}{x}\right)\right] &= \frac{y}{\sqrt{x^2+y^2}} \end{aligned}$$

The curl or $\nabla\times$ in 3D is given by

$$\nabla \times \vec{u} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$\nabla \times \vec{u} = \hat{x}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) - \hat{y}\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) + \hat{z}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{1}$$

(a) Solid body rotation: $u_r = 0$, $u_\theta = \omega r$

$$\begin{aligned} \vec{u} &= u_r\hat{r} + u_\theta\hat{\theta} \\ &= \omega\sqrt{x^2+y^2}\frac{-y\hat{x}+x\hat{y}}{\sqrt{x^2+y^2}} \\ &= \omega(-y\hat{x}+x\hat{y}) \end{aligned}$$

such that the velocity components in the $x-$ and $y-$directions are given by $u = -\omega y$ and $v = \omega x$ respectively. Figure 1a shows the velocity vector as function of (x, y) for this example.
The vorticity is given by $\nabla \times \vec{u}$ such that

$$\nabla \times \vec{u} = \hat{z}\left(\frac{\partial}{\partial x}(\omega x) - \frac{\partial}{\partial y}(-\omega y)\right) = 2\omega\hat{z} \tag{2}$$

The vorticity has only a vertical component and is equal to twice the angular velocity, similar to the results obtained in class using cylindrical coordinates.

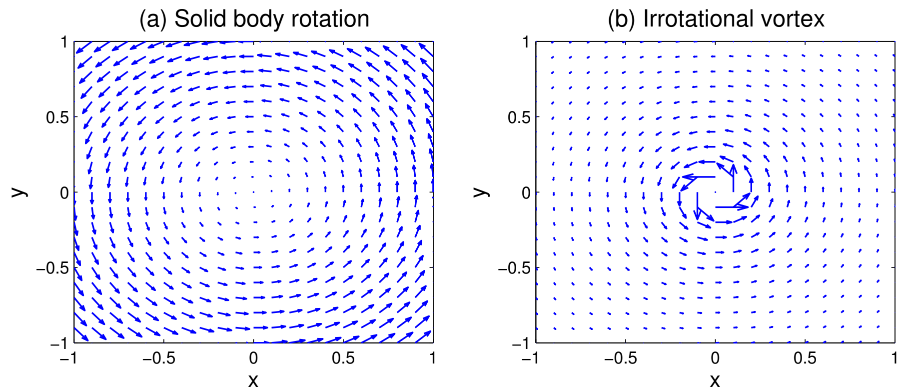

Figure 1: Velocity vectors in the $x-y$ plane using quiver for (a) solid body rotation and (b) irrotational vortex.

(b) Irrotational vortex: $u_r = 0$, $u_\theta = \Lambda/2\pi r$

$$\begin{aligned}\vec{u} &= u_r\hat{r} + u_\theta\hat{\theta} \\ &= \frac{\Lambda}{2\pi\sqrt{x^2+y^2}}\frac{-y\hat{x}+x\hat{y}}{\sqrt{x^2+y^2}} \\ &= \frac{\Lambda}{2\pi}\left(\frac{-y\hat{x}+x\hat{y}}{x^2+y^2}\right)\end{aligned}$$

Figure 1b shows the velocity vector as function of (x,y). The vorticity is

$$\nabla\times\vec{u} = \hat{z}\frac{\Lambda}{2\pi}\left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right)\right] = 0 \tag{3}$$

The vorticity is equal to zero. By definition, the vorticity is a measure of the rotation of the flow, in this case the vorticity is zero therefore the name irrotational.

(c) Exponential velocity: $u_r = 0$, $u_\theta = a\,e^{-r/r_0}$

$$\begin{aligned}\vec{u} &= u_r\hat{r} + u_\theta\hat{\theta} \\ &= ae^{-\sqrt{x^2+y^2}/r_0}\frac{-y\hat{x}+x\hat{y}}{\sqrt{x^2+y^2}} \\ &= -\frac{aye^{-\sqrt{x^2+y^2}/r_0}}{\sqrt{x^2+y^2}}\hat{x} + \frac{axe^{-\sqrt{x^2+y^2}/r_0}}{\sqrt{x^2+y^2}}\hat{y}\end{aligned}$$

The vorticity is then

$$\begin{aligned}\nabla\times\vec{u} &= \hat{z}\left[\frac{\partial}{\partial x}\left(\frac{axe^{-\sqrt{x^2+y^2}/r_0}}{\sqrt{x^2+y^2}}\right) - \frac{\partial}{\partial y}\left(-\frac{aye^{-\sqrt{x^2+y^2}/r_0}}{\sqrt{x^2+y^2}}\right)\right] \\ &= a\frac{r_0-\sqrt{x^2+y^2}}{r_0\sqrt{x^2+y^2}}e^{-\sqrt{x^2+y^2}/r_0}\hat{z}\end{aligned}$$

Figure 2 shows the velocity field in the $x-y$ plane and contours of vorticity. As expected from our solutions, the velocity and vorticity are infinite at the origin and decay away from it. The rotation is counterclockwise and therefore the vorticity is positive.

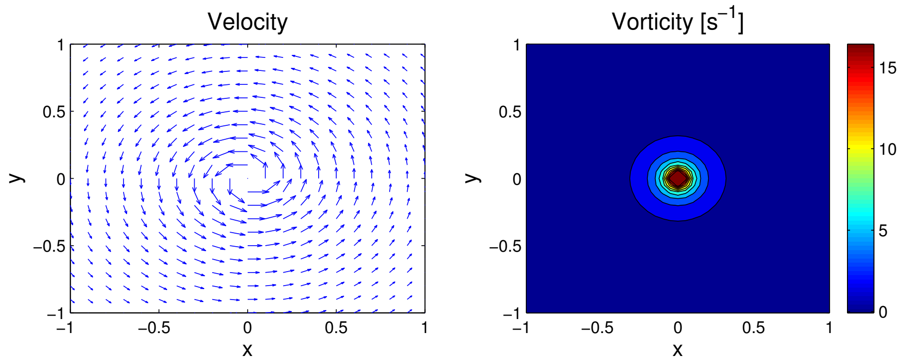

Figure 2: Velocity vectors in the $x-y$ plane using quiver and contours of the vorticity field.

2. Wind stress and Sverdrup relation: Assume that the wind stress is given by

$$\vec{\tau} = (\tau^{(x)}, \tau^{(y)}) = (\tau_0 \cos[\frac{\pi}{20}(40-\theta)], 0) \tag{4}$$

where $\tau_0 = 0.7 dyn/cm^2 = 0.07 N/m^2$ and $20N \leq \theta \leq 60N$.

(a) The wind stress as function of latitude is shown in Fig. 3.

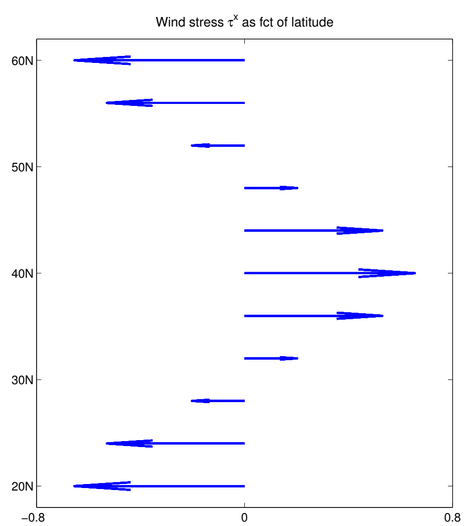

Figure 3: Wind stress as function of latitude.

(b) North-South velocity in the interior from the Sverdrup relation: the Sverdrup relation is a balance between the north-south advection of planetary vorticity and the Ekman pumping at the base of the Ekman layer. This balance is valid in the ocean interior and away from the western boundaries. The Sverdrup relation is given by

$$\beta v = \frac{1}{\rho_0 H} \nabla \times \vec{\tau}^{wind} \tag{5}$$

The wind stress is a function of θ, using the fact that we have approximatively

$$1^\circ\, lat = \frac{1}{360}\,(2\pi R) \approx 111km$$

we can convert all the angles to meters such that the domain defined by $20N \leq \theta \leq 60N$ is equivalent to $222\cdot 10^4\, m \leq y \leq 666\cdot 10^4\, m$, such that

$$\vec{\tau} = (\tau^{(x)}, \tau^{(y)}) = (\tau_0 \cos[\frac{\pi}{222\cdot 10^4}(444\cdot 10^4 - y)], 0) \tag{6}$$

The curl of the wind stress is given by

$$\begin{aligned}\nabla \times \vec{\tau} &= -\frac{\partial}{\partial y}\left(\tau_0 \cos[\frac{\pi}{222\cdot 10^4}(444\cdot 10^4 - y)]\right)\\ &= -\frac{\tau_0 \pi}{222\cdot 10^4} sin[\frac{\pi}{222\cdot 10^4}(444\cdot 10^4 - y)]\end{aligned}$$

It is easy now to write an expression for the north-south velocity such that

$$v = -\frac{\tau_0 \pi}{\beta \rho_0 H(222\cdot 10^4)} sin[\frac{\pi}{222\cdot 10^4}(444\cdot 10^4 - y)] \tag{7}$$

Note: another way to do this will be to use the chain rule such that

$$\begin{aligned}\frac{\partial \tau^{(x)}}{\partial y} &= \frac{\partial \tau^{(x)}}{\partial \theta}\frac{\partial \theta}{\partial y}\\ &= \frac{9\tau_0}{R}\sin[\frac{\pi}{20}(40-\theta)]\end{aligned}$$

where $\theta = 360\cdot y/(2\pi R) \Rightarrow \partial\theta/\partial y = 180/(\pi R)$ such that

$$v = -\frac{9\tau_0}{\rho_0 H \beta R} sin[\frac{\pi}{20}(40-\theta)] \tag{8}$$

Obviously you get the exact same answer.

(c) Volume transport:

$$\begin{aligned}M_{sverdrup} &= H\Delta x v\\ &= -\frac{\tau_0 \pi}{\beta \rho_0 H(222\cdot 10^4)} sin[\frac{\pi}{222\cdot 10^4}(444\cdot 10^4 - y)]\end{aligned}$$

Figure 4 shows the velocity and the Sverdrup transport as function of latitude. We see that the velocity and the transport are antisymmetric about $40^\circ N$, where the curl of the wind stress vanishes.

(d) In the Sverdrup interior, the velocity and therefore the transport are proportional to the curl of the wind stress. Between $20^\circ N$ and $40^\circ N$, the curl of the wind stress is clockwise and add negative vorticity to the ocean between those latitudes, the flow will therefore moves to the south. The velocity and transport are negative between the latitudes $20^\circ N$ and $40^\circ N$ consistent with the southward velocity in the subtropical gyre. The values for the maximum velocity in the interior are roughly $0.4 cm/s$ and the transport close to $30 Sv$ consistent with observations. Between $40^\circ N$ and $60^\circ N$, the curl of the wind stress is counterclockwise and therefore add positive vorticity to the ocean between those latitudes and the flow is northward. This picture is consistent with the existence of the subpolar gyre in the Atlantic and the Pacific oceans.

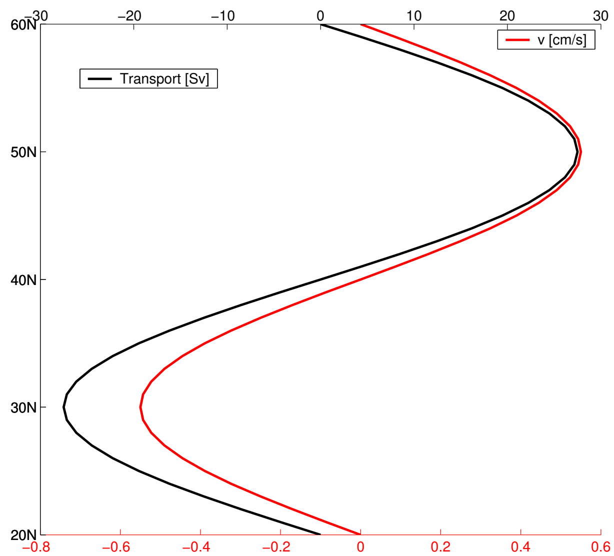

Figure 4: Velocity in cm/s and volume transport in Sv in the interior as function of latitude using $\beta = 2\Omega cos(40°N)/R \approx 1.76 \cdot 10^{-11}(m \cdot s)^{-1}$, $\rho_0 = 1026 kg/m^3$, $\tau_0 = 0.07 N/m^2$, $H = 1000m$, $W = 5000km$ and $R = 6371km$.