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Introduction to Physical Oceanography
Homework 4 - Solutions

1. Geostrophy and thermal wind

   (a) The satellite observed a sea surface height increase of $1m$ over $200km$ toward the southeast at $35°N$. The geostrophic velocity at the surface (magnitude and direction) is given by

$$f\hat{k} \times \vec{u} = -\frac{1}{\rho_0}\nabla p \tag{1}$$

   Using the vertically integrated equation for the hydrostatic pressure $p = \rho_0 g\,(h(x,y) - z))$, we can write

$$\hat{k} \times \vec{u} = -\frac{g}{f}\nabla h \tag{2}$$

   The magnitude of the geostrophic velocity at the surface is then given by

$$|\vec{u}| = \frac{g}{f}|\nabla h| = \frac{9.81m/s^2}{2 \cdot 7.3 \cdot 10^{-5} sin(35°N) s^{-1}} \frac{1m}{200km} = 0.5857m \cdot s^{-1} \tag{3}$$

   The slope of the sea surface increases toward the southeast, therefore the pressure gradient force is directed from the southeast to northwest, and to balance this force Coriolis is in the opposite direction. Since in the northern hemisphere, Coriolis acts to the right of the velocity, we conclude that the direction of the geostrophic velocity at the surface is northeast (see figure 1).

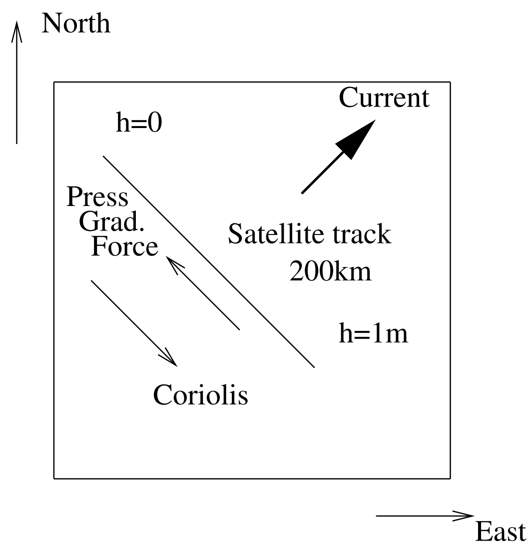

Figure 1: Forces and current for question 1(a).

(b) The average density for the 2 stations A and B are from a depth of 2000db to the surface. We assume that the level $p_0 = 2000db$ is flat meaning that $z_A = z_B = z(p_0 = 2000db)$ (this level is called a level of no motion, where at a given depth z, the horizontal pressure gradient vanishes and the geostrophic velocity is therefore zero). Integrating the hydrostatic equation from $p_0 = 2000db$ up to the surface, we obtain for station A

$$\frac{1}{\rho_A g}(p_{atm} - p_0) = -h_A + z(p_0 = 2000db) \tag{4}$$

and for station B

$$\frac{1}{\rho_B g}(p_{atm} - p_0) = -h_B + z(p_0 = 2000db) \tag{5}$$

Taking the difference between these 2 equations and assuming that $p_{atm} = 0$:

$$\frac{p_0}{g}\left(\frac{1}{\rho_B} - \frac{1}{\rho_A}\right) = -h_A + h_B \tag{6}$$

Using the values $g = 9.81m/s^2$, $p_0 = 2000db$, $\rho_A = 1027kg/m^3$ and $\rho_B = 1027.8kg/m^3$, the in the sea surface height difference is $\Delta h = h_B - h_A = -1.55m$. The sea surface height increases from station B to station A by $1.55m$ over a distance of $L = 250km$, the slope of the sea surface height α is given by

$$\alpha = \frac{\Delta h}{L} = 6.2 * 10^{-6} \tag{7}$$

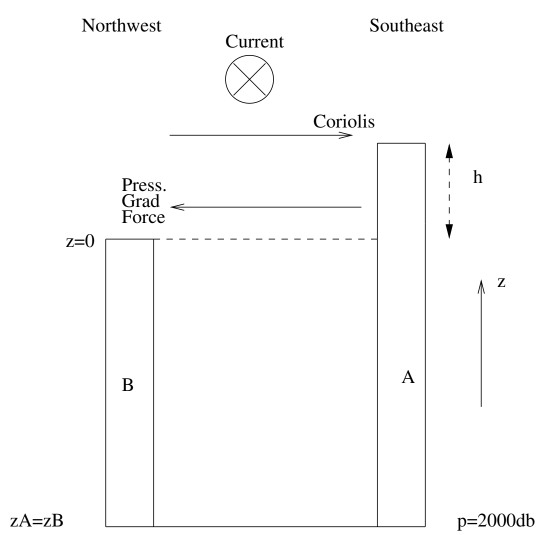

Figure 2: The water columns at station A and B showing the level of no motion at $p_0 = 2000db$, the balance of forces and the surface current direction.

Given the slope of the sea surface, we can find the magnitude and the direction of the current using the same expression than the one found in 1(a):

$$|\vec{u}| = \frac{g}{f}|\nabla h| = \frac{9.81m/s^2}{2 \cdot 7.3 \cdot 10^{-5} sin(35°N)s^{-1}} \cdot 6.2 * 10^{-6} = 0.72m \cdot s^{-1} \tag{8}$$

From figure 2 it is clear that the direction is North-East (same than the direction given by figure 1).

(c) In this question, the sea surface height difference between station A and B is known and equal to

$$\Delta h = h_A - h_B = \frac{1m}{200km} 250km = 1.25m \tag{9}$$

In addition, the average density from a level of 2000db up to the surface is $\rho_A = 1027 kg/m^3$ and $\rho_B = 1027.8 kg/m^3$ for station A and B respectively. Since the level of 2000db is not a level of no motion anymore, it means that the pressure surface has a slope (not flat anymore) or equivalently that $z_A(p = 2000db) \neq z_B(p = 2000db)$. We can again integrate the hydrostatic equation from $p = 2000db$ up to the surface such that at station A, we obtain

$$\frac{1}{\rho_A g} (p_{atm} - 2000db) = -h_A + z_A(p = 2000db) \tag{10}$$

and at station B

$$\frac{1}{\rho_B g} (p_{atm} - 2000db) = -h_B + z_B(p = 2000db) \tag{11}$$

Again assuming that $p_{atm} = 0$ and subtracting the 2 equations:

$$\frac{2000db}{g} \left(\frac{1}{\rho_B} - \frac{1}{\rho_A} \right) = -h_A + h_B + z_A(p = 2000db) - z_B(p = 2000db) \tag{12}$$

leading to

$$z_A(p = 2000db) - z_B(p = 2000db) = \frac{2000db}{g} \left(\frac{1}{\rho_B} - \frac{1}{\rho_A} \right) + h_A - h_B \tag{13}$$

Using $\Delta h = h_A - h_B = 1.25m$ and $1db = 10^4 Pa$,

$$z_A(p = 2000db) - z_B(p = 2000db) = -0.2952m \tag{14}$$

The level $p = 2000db$ slopes toward the northwest. Therefore at a given depth near the level $p = 2000db$, the pressure at the station A is smaller than the pressure at the station B. At station B for a depth equal to z_B the pressure is equal to 2000db, let's find what is the pressure at station A at the same depth using $z_A = z_B - 0.2952m$ and the hydrostatic equation:

$$\begin{aligned} p_A(z_B) &= \rho_A g (h_A - z_A + 0.2952m) \\ &= \rho_A g (h_A - z_A) - \rho_A g \cdot 0.2952m \\ &= 2000db - \rho_A g \cdot 0.2952m \approx 1999.7025db \end{aligned}$$

As expected at a given depth $z = z_B$, the pressure at station A is smaller than the pressure at station B and this difference of pressure is approximatively equal to $0.3db$ over a distance of $250km$, such that $|\nabla p| = 0.3db/250km$. Finally, using the geostrophic equation, we can find the magnitude of the velocity at a depth of 2000db:

$$|\vec{u}| = \frac{1}{f\rho} |\nabla p| \approx 0.138 m \cdot s^{-1} \tag{15}$$

To find the direction of the current at this depth: the pressure gradient is southeast and the Coriolis force is directed northwest, the velocity near 2000db is then southwest (see Fig. 3.

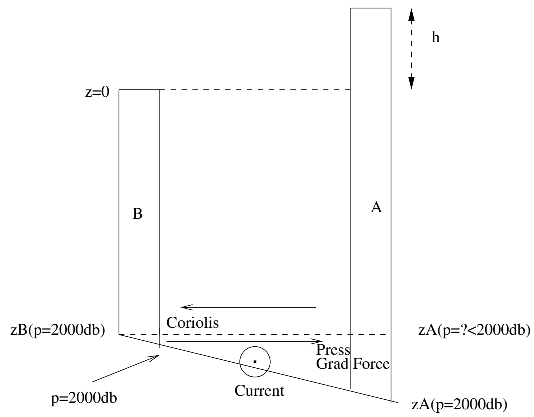

Figure 3: The water columns at station A and B showing the $p = 2000db$ level (not flat anymore) and deep current direction.

(d) Assume that stations A and B have the same salinity. The column of water at station B is shorter and denser than the one at station A, we can expect the water at station A to be warmer than the water at station B. At the surface, if we look downstream at a point midway between A and B, the warm water is the right of the current. But at a depth of 2000db, if we look downstream at a point midway between A and B, the warm water is the left of the current.

2. T-S diagram and water masses mixing:

(a) This station is located in the South Atlantic between the mid-Atlantic ridge and the coast of Angola.

(b) See Fig. 4

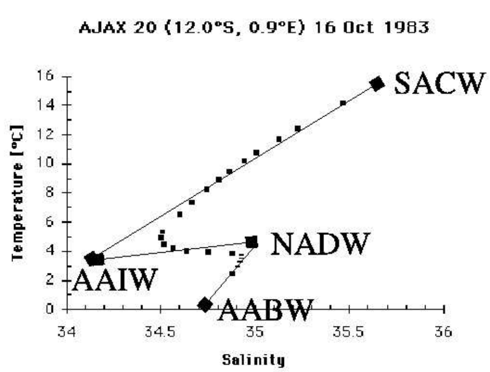

Figure 4: T-S diagram and the different water masses

(c) The data on the T-S diagram fit perfectly well the idea of water masses being formed by mixing of waters from different water types.

(d) The water masses are:

- North Atlantic Deep water (NADW): $3 < T < 5°C$, $34.9 < S < 35ppt$. It extends at depth of about 2km. We think that the source of the North Atlantic Deep water is principally Greenland (about 80% of the deep water is formed there). The formation of deep water occurs by open-ocean process (deep water is formed from the surface by cooling and then sinks and overflows the sills between Greenland and Scotland into the North Atlantic). Some of NADW is formed in the Labrador Sea by the same process and occasionally in the Irminger Sea.
- Antarctic Bottom Water (AABW): $-1.9 < T < 0°C$, $34.6 < S < 34.7ppt$. It is the densest water mass of the World Ocean! AABW is found to occupy the depth range below 4000 m of all ocean basins that have a connection to the Southern Ocean at that level (to enter the oceans, the AABW has to pass through and mix with the water of the Antarctic Circumpolar Current). Most of it is formed in the Weddell Sea and Ross Sea by deep winter convection (along the coast of Antarctica). The formation of the AABW is a near-boundary sinking occurring due to the formation of ice. While ice freezes out, it rejects salt and increase the density of the water (which was already cold and dense). This water sinks along the shelf and down the slope into the South Atlantic.
- Antarctic Intermediate Water (AAIW): $3 < T < 4°C$, $34.1 < S < 34.2ppt$. This water mass has a salinity minimum found at depths between 700 and 1000 m in the Southern Hemisphere. It is formed at various locations along the Antarctic Polar Front and through deep winter convection east of southern Chile and south of the Great Australian Bight. It enters all oceans with the Antarctic Circumpolar Current and spreads toward the equator between the central water and the deep water.
- South Atlantic Central Water (SACW): $4 < T < 16°C$, $34.2 < S < 35.6ppt$. In general, central waters are water masses that form directly above the permanent thermocline. Generally, they are confined to regions closer to the equator and near the surface. This water mass is south of $15°N$ with pretty uniform properties throughout its range and can be represented by a straight line on the $T-S$ diagram. This water is subducted from the Subtropical Convergence Zone.

3. Buoyancy oscillations and equation of state: from the table posted on the course home page, we see that the thermal expansion coefficient ($\alpha = (1/\rho_0)\partial\rho/\partial T|_{T_0,S_0}$) at a given pressure p is highly dependent on temperature while the haline expansion coefficient $(1/\rho_0)\partial\rho/\partial S|_{T_0,S_0}$ varies very little with temperature.

   The equation of state for the ocean is a very complicated nonlinear equation and as a first approximation we simply do a Taylor expansion of $\rho(T,S)$ at a given pressure around some reference density ρ_0 such that $\rho_0 = \rho_0(T_0,S_0)$.

   Consider a given temperature profile (salinity is assumed to be uniform and equal to $35ppt$):

   CASE 1: the temperature changes from $3°C$ at the surface to $1°C$ at a depth of $200m$. At this depth the pressure is roughly equal to 20bar, for our purpose, we will assume that the values of α and β at $200m$ are similar to those at the surface, therefore at both depth α and β are given for a pressure $p = 0$ from the table. For this case, we will linearize the equation of state around $T_0 = 2°C$ and $S_0 = 35ppt$ such that

$$\rho(T,S) = \rho_0(T_0 = 2, S_0 = 35)\left(1 - \alpha(T-2) + \beta(S-35)\right) \tag{16}$$

Since we assumed that the salinity is uniform, the last term in this equation is zero leading to

$$\rho(T,S) = \rho_0(T_0 = 2, S_0 = 35)\,(1 - \alpha(T - 2)) \tag{17}$$

From the table, $\rho_0 = 1027.972 kg/m^3$ and $\alpha = 781 * 10^{-7} K^{-1}$, and figure 5.a shows the density profile as function of temperature (of course, the profile is linear).

The Brunt-Vaisala frequency N is the frequency of vertical oscillations in a stratified fluid (we obtained this frequency from a balance between the vertical acceleration and the buoyancy in the z-momentum equation). The square of the Brunt-Vaisala frequency is given by

$$N^2 = -\frac{g}{\rho_0}\frac{\partial \rho}{\partial z} \tag{18}$$

Using the linear equation of state, the chain rule ($\frac{\partial \rho}{\partial z} = \frac{\partial \rho}{\partial T}\frac{\partial T}{\partial z}$) and the definition of α, we obtain

$$N^2 = -g\alpha\frac{\partial T}{\partial z} \tag{19}$$

For this case, we obtain that $N = 0.0028 sec^{-1}$. The period of oscillation is given by $T = 2\pi/N = 2270 sec = 0.6306 hr$.

CASE 2: the temperature changes from $23°C$ at the surface to $21°C$ at a depth of $200m$. We use the same arguments than those mentioned above, but we will now linearize around $T_0 = 22°C$.

$$\rho(T,S) = \rho_0(T_0 = 22, S_0 = 35)\,(1 - \alpha(T - 22)) \tag{20}$$

From the table, at this temperature we have $\rho_0 = 1024.219 kg/m^3$ and $\alpha = 2734 K^{-1}$. Figure 5.b shows the density profile as function of temperature.

For this case, we obtain that $N = 0.0052 sec^{-1}$. The period of oscillation is given by $T = 2\pi/N = 1213.2 sec = 0.337 hr$.

Conclusion: we started with 2 different cases: case 1 had colder water at the surface than case 2. But, the vertical gradient of temperature of case 1 was equal to the vertical gradient of temperature of case 2. Even though the vertical temperature gradient is equal in both cases, due to dependency of the thermal expansion coefficient on temperature, their vertical density gradient differ (we can see the slope on the right plot in Fig. 5. In addition, considering only the effect of temperature in the ocean, we can see from the figure above that dependent around which point we linearized the equation of state, we could get up to $4 kg/m^3$ difference in density at the surface. The first order Taylor expansion is a good approximation in general, but one has to keep in mind that due to the strong dependence of α on the temperature it is not a great approximation.

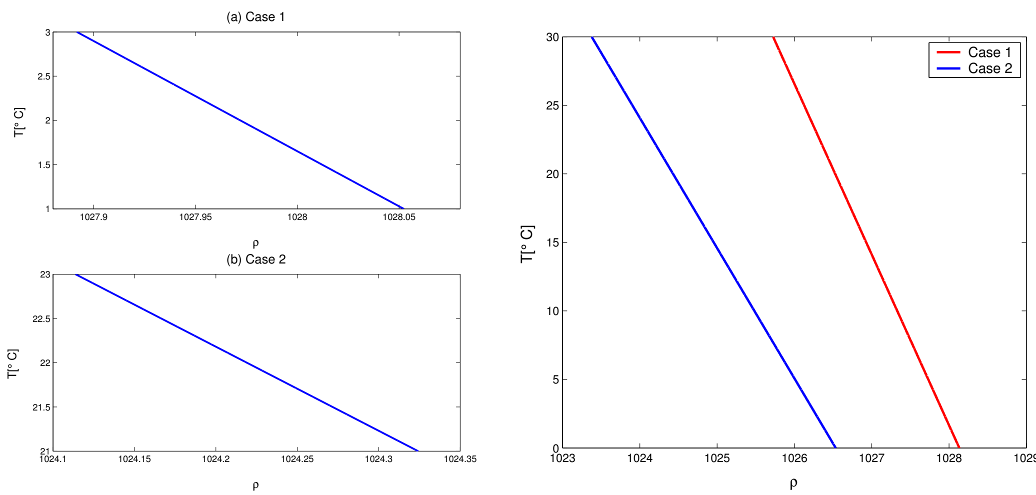

Figure 5: On the left: Density profile as function of temperature (a) for case 1 where T varies from $3°C$ to $1°C$, (b) for case 1 where T varies from $23°C$ to $21°C$; on the right: Density profile as function of temperature for Case 1 and 2 extrapolated for T varying from $0°C$ to $30°C$.