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10/11/05

Introduction to Physical Oceanography
Homework 2 - Solutions

1. Abyssal recipes

   (a) Figure 1 shows the vertical temperature profile at two different locations in the Pacific ocean, (a) near the equator at $(0.5°S, 179.5°E)$; (b) at $(45.5°N, 160.5W)$.

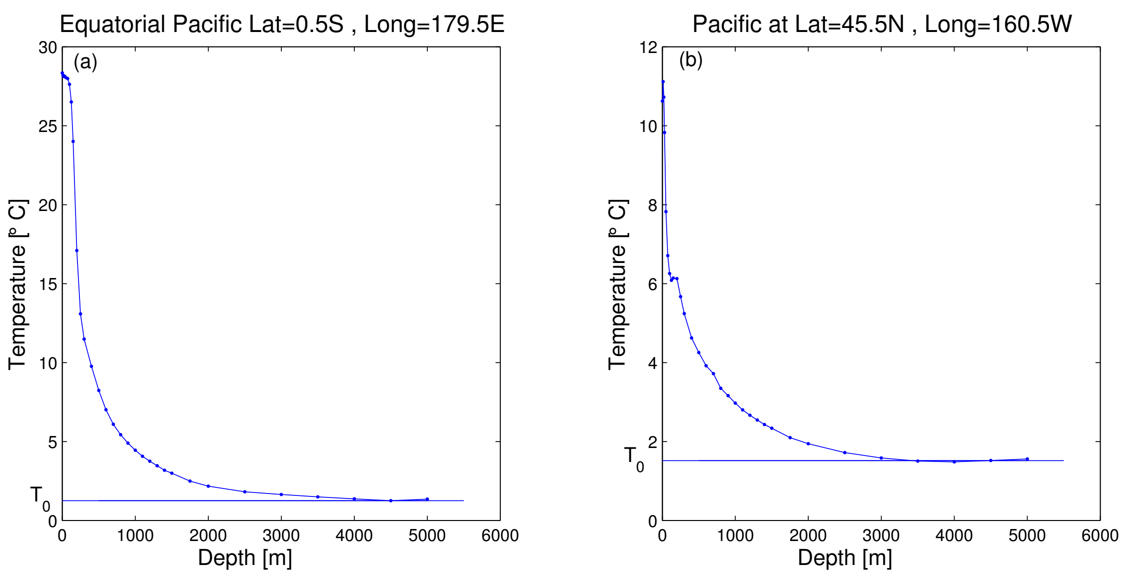

Figure 1: Vertical temperature profile of the Pacific ocean (a) at $(0.5°S, 179.5°E)$; (b) at $(45.5°N, 160.5W)$.

   (b) I will describe two different way to fit the vertical temperature profile from a depth of $500m$ to the bottom of the ocean.

   Matlab has a very nice built-in function to perform linear Least Squares Fitting. If you are not familiar with least squares fitting, you might want to check
   "http://mathworld.wolfram.com/LeastSquaresFitting.html"
   for a short description of the method. We wish to fit the temperature from a depth of 500m to the bottom of the ocean using an exponential function of the form $T(z) = T_0 + a \cdot e^{-z/H}$. The values of T_0 are found graphically (see Figure 1). My Matlab code as well as the data I used are posted on the course website if you wish to follow the procedure I used to fit this data. The results obtained for H are the following: at the equator, $H = 704.2254m$, while at $45°N$ $H = 744.6016m$. The results for this fit are shown in Figure 2.

   Another way to fit the data, less rigorous than least squares fit but good enough as a first approximation is described here. We can evaluate T_0 as previously mentioned from the graph: at the equator $T_0 = 1.2585°C$ and at $45°N$ $T_0 = 1.5170°C$.

   We wish to fit the data to exponential function of the form $T(z) = T_0 + a \cdot exp(z/H)$. From this data set, the temperature at a depth of $500m$ is found to be equal to: $T(z =$

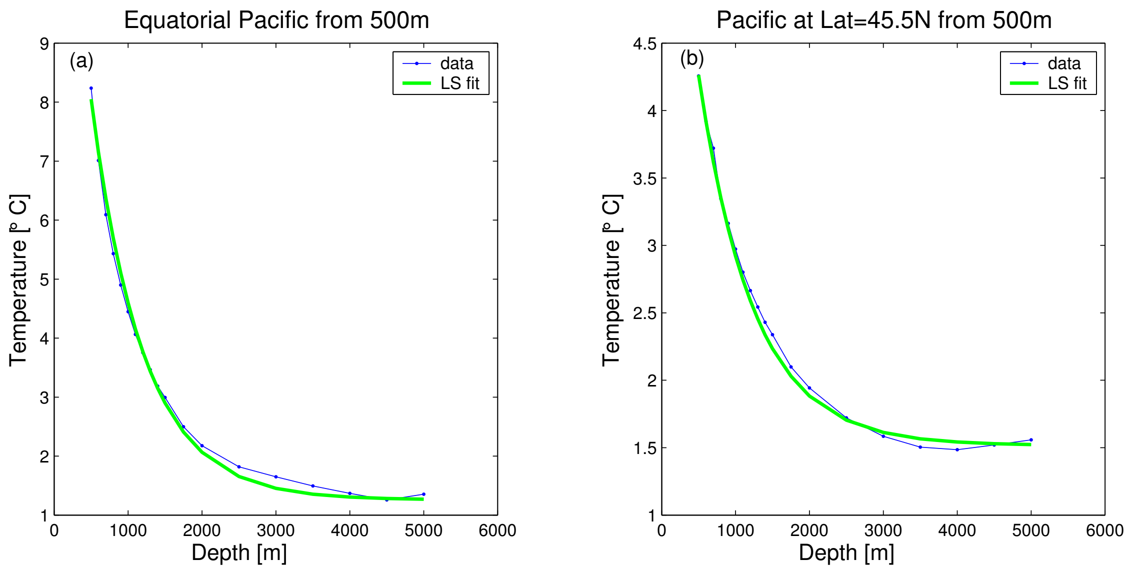

Figure 2: Vertical temperature profile of the Pacific ocean from the data (blue line) and using a least squares fit (green line) (a) at the equator; (b) at 45°N.

$500m) = 8.2359°C$ at the equator and and to $T(z = 500m) = 4.2564°C$ at 45°N. Using these values as well as the values for T_0 we get: $a \cdot exp(500/H) = 6.9774$ at the equator; and $a * exp(-500/H) = 2.7394$ at 45°N.

We still need to find the coefficients a and H. Let find the temperature at a depth of $z = 500 + H$, such that

$$T(500+H) = T_0 + a \cdot e^{-\frac{500+H}{H}} = T_0 + a \cdot e^{-\frac{500}{H}} \cdot e^{-1} \tag{1}$$

Using the values for $a \cdot e^{-\frac{500}{H}}$ found previously, we get $T(500+H) = 3.8253°C$ at the equator and $T(500+H) = 2.5248 \circ C$ at 45°N. From our plots, we find that these temperatures correspond to a depth of approximatively 1200m at the equator, and to a depth of 1300m at 45°N. This leads to values for H equal to $H = 700m$ and $H = 800m$ at the equator and at 45°N respectively. Finding a is now pretty easy: at the equator $a = 14.2529$ and $a = 5.1179$ at 45°N. Figure 3 shows the results for this fit…not too bad!

In class, we derived an equation for the temperature. We made several different assumptions: the density of the ocean is nearly constant, the temperature of the ocean is in a steady state ($\partial T/\partial t = 0$), the temperature is only a function of z and finally we assumed that no heating or cooling from the atmosphere can affect the temperature at a depth below 500m. From these assumptions, we found that the vertical equation for the temperature results in a balance between vertical eddy mixing and vertical advection such that

$$w\frac{\partial T}{\partial z} = \kappa \frac{\partial^2 T}{\partial z^2} \tag{2}$$

leading to the following scaling for κ

$$\kappa = H w \tag{3}$$

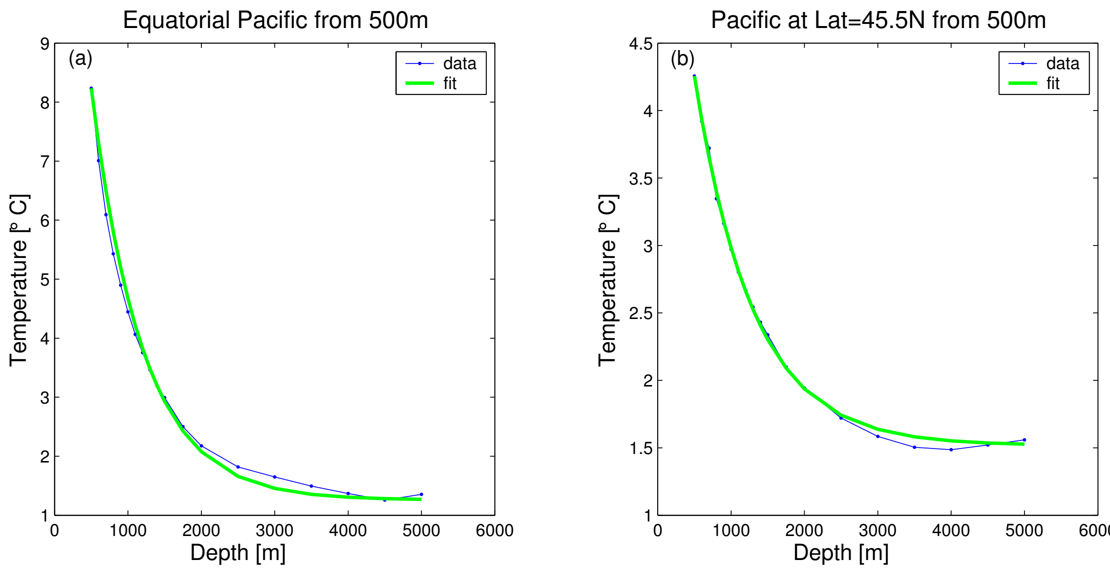

Figure 3: Vertical temperature profile of the Pacific ocean from the data (blue line) and using the 2nd fit (green line) (a) at the equator; (b) at $45°N$.

Using the value $w = 10^{-4} cm \cdot s^{-1}$ and those obtained for H, I found $\kappa = 7 \cdot 10^{-4} m^2 \cdot s^{-1} \approx 7 cm^2 \cdot s^{-1}$ at the equator and $\kappa = 8 \cdot 10^{-4} m^2 .s^{-1} \approx 8 cm^2 .s^{-1}$ at $45°N$.

For a given w, we found different values for κ at 2 different locations in the Pacific ocean. Therefore for a constant vertical velocity, we cannot fit our profiles with similar coefficients for eddy mixing. κ is a function of location, it depends on depth, topography... depending on different physical mechanisms taking place in the ocean. In addition, w is also not uniform in the ocean and varies as function of location.

2. Consider an accelerating fluid parcel flowing along the center of slowly narrowing channel. The velocity is given by

$$\vec{u} = (u(x), 0, 0). \tag{4}$$

(a) The width of the channel is $W = W_0/x$, we assume that the depth H is constant and the volume flux F is constant along the channel such that

$$\begin{aligned} F &= velocity \cdot Area = constant \\ F &= u(x) \cdot W \cdot H = W_0 H \frac{u(x)}{x}, \end{aligned}$$

the Eulerian velocity is therefore given by

$$u(x) = \frac{F \cdot x}{W_0 H}. \tag{5}$$

The Eulerian velocity is simply linearly increasing with x (and does not depend on time). This results is physically consistent with the fact that the width of the channel is proportional to $1/x$.

(b) To find the Lagrangian location $x(t;x_0)$ of a fluid parcel which was at x_0 at $t=0$, we need to integrate the velocity field

$$\begin{aligned} \frac{dx}{dt} &= u(x) = \frac{F\cdot x}{W_0 H} \\ \int_{x_0}^{x} \frac{dx}{x} &= \int_0^t \frac{F}{W_0 H} dt \end{aligned}$$

The result of this integration leads to the Lagrangian trajectory

$$x(t;x_0) = x_0 e^{\frac{F}{W_0 H}t} \tag{6}$$

(c) Now that the Lagrangian trajectory was found, the Lagrangian velocity is given by

$$u(t;x_0) = \frac{dx(t;x_0)}{dt} = \frac{F}{W_0 H} x_0 e^{\frac{F}{W_0 H}t} \tag{7}$$

and the Lagrangian acceleration by

$$a(t;x_0) = \frac{du(t;x_0)}{dt} = \left(\frac{F}{W_0 H}\right)^2 x_0 e^{\frac{F}{W_0 H}t} \tag{8}$$

(d) In this question, we are asked to find the acceleration by taking the material derivative of the Eulerian velocity

$$\begin{aligned} a(x,t) &= \frac{du(x,t)}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} \\ a(x,t) &= \frac{\partial}{\partial t}\left(\frac{F\cdot x}{W_0 H}\right) + \left(\frac{F\cdot x}{W_0 H}\right)\frac{\partial}{\partial x}\left(\frac{F\cdot x}{W_0 H}\right) \end{aligned}$$

Since the Eulerian velocity is steady ($\frac{\partial u}{\partial t} = 0$, the local rate of change of the velocity at a fixed location is 0), the acceleration is then

$$a(x,t) = \left(\frac{F}{W_0 H}\right)^2 x \tag{9}$$

If we substitute $x = x_0 e^{\frac{F}{W_0 H}t}$ in the last expression for $a(x,t)$, we see that the 2 results agree.

3. Challenge Problem: the Eulerian velocity field is given by

$$\begin{aligned} u_1 &= U_0 \\ u_2 &= V_0 cos\left(k\left[x_1 - ct\right]\right) \\ u_3 &= z_0 \end{aligned}$$

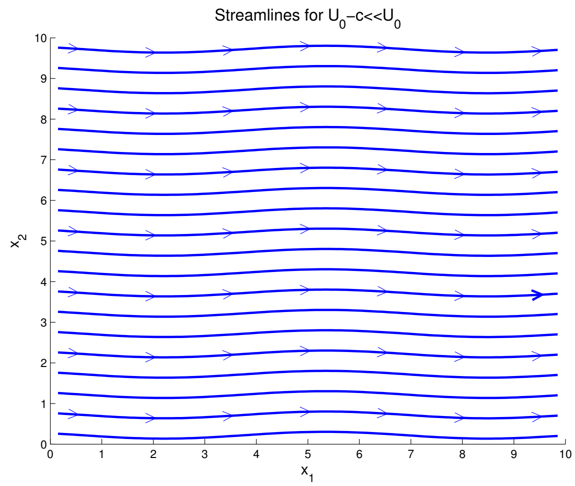

Figure 4: Streamlines of the flow at t=10.

(a) Streamlines of the flow (lines tangent to the velocity at a given time)

$$\begin{aligned} \frac{dx_2}{dx_1} &= \frac{u_2}{u_1} = \frac{V_0 cos\,(k\,[x_1 - ct])}{U_0} \\ \int^{x_2} dx_2 &= \frac{V_0}{U_0} \int^{x_1} cos\,(k\,[x_1 - ct])\,dx_1 \end{aligned}$$

leading to

$$x_2 = \frac{V_0}{U_0 k} sin\,(k\,[x_1 - ct]) + C \tag{10}$$

The streamlines are shown in Figure 4 for $t = 10$.

(b) Trajectory of a fluid element which was at (X_1, X_2, X_3) at $t = 0$.
For x_3 it is pretty straightforward:

$$x_3 = X_3 \tag{11}$$

To find the trajectory of $x_1(t; X_1)$ and $x_2(t; X_2)$, we must integrate the velocity field:

$$\frac{dx_1}{dt} = u_1 = U_0 \Rightarrow \int_{X_1}^{x_1} dx_1 = \int_0^t U_0 dt \tag{12}$$

The trajectory of the parcel in the x_1 direction is given by

$$x_1 = U_0 t + X_1 \tag{13}$$

We can repeat the same procedure to find $x_2(t; X_2)$

$$\frac{dx_2}{dt} = u_2 = V_0 cos\,(k\,[x_1 - ct]) \tag{14}$$

Using our results for $x_1(t; X_1)$,

$$\begin{aligned} \frac{dx_2}{dt} &= V_0 cos\,(k\,[X_1 + (U_0 - c)t]) \Rightarrow \\ \int_{X_2}^{x_2} dx_2 &= \int_0^t V_0 cos\,(k\,[X_1 + (U_0 - c)t])\,dt \end{aligned}$$

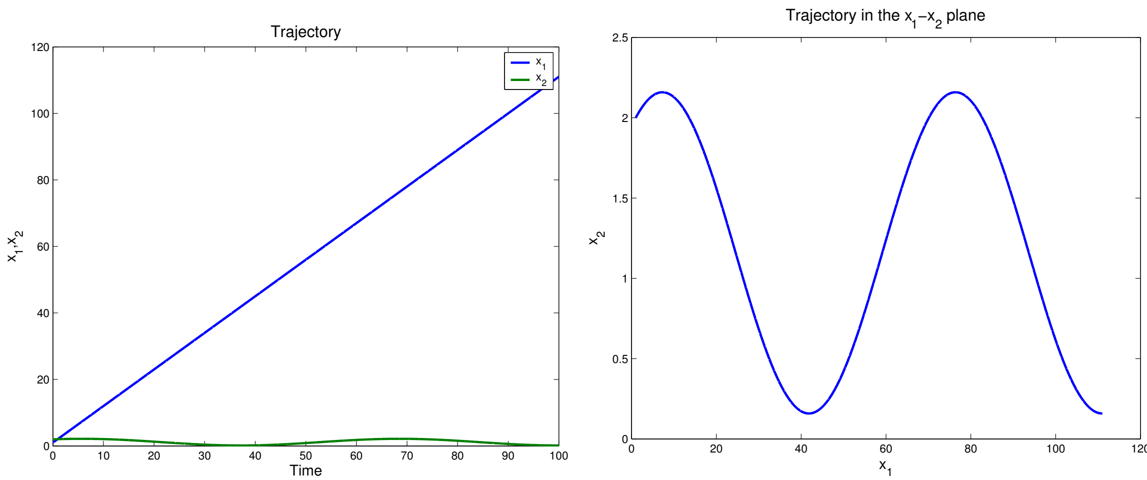

Figure 5: Lagrangian trajectory

We obtain

$$
\begin{aligned}
x_2 &= X_2 + \frac{V_0}{k(U_0-c)}\left[sin\left[k((U_0-c)t+X_1)\right] - sin(kX_1)\right] \\
x_2 &= \frac{V_0}{k(U_0-c)} sin\left[k((U_0-c)t+X_1)\right] + X_2 - \frac{V_0}{k(U_0-c)} sin(kX_1)
\end{aligned}
$$

To summarize, the trajectory of a fluid element which was at (X_1, X_2, X_3) at $t = 0$ is given by

$$
\begin{aligned}
x_1 &= U_0 t + X_1 \\
x_2 &= \frac{V_0}{k(U_0-c)} sin\left[k((U_0-c)t+X_1)\right] + X_2 - \frac{V_0}{k(U_0-c)} sin(kX_1) \\
x_3 &= X_3
\end{aligned}
$$

Figure 5 shows the trajectory as a time series for x_1 and x_2 as function of time as well as the trajectory of the fluid element in the $x_1 - x_2$ plane.

(c) Wavelength of the Eulerian streamlines:

The wavelength is the distance between repeating points of a periodic wave (for example, the distance between two consecutive crests) such that at any given time τ we have $x_2(x_1+\lambda) = x_2(x_1)$.

From our previous results, we can write

$$
\begin{aligned}
x_2(x_1,\tau) &= \frac{V_0}{U_0 k} sin\left(k\left[x_1 - c\tau\right]\right) + C = \\
x_2(x_1+\lambda,\tau) &= \frac{V_0}{U_0 k} sin\left(k\left[x_1 + \lambda - c\tau\right]\right) + C \\
x_2(x_1+\lambda,\tau) &= \frac{V_0}{U_0 k} sin\left(kx_1 + k\lambda - kc\tau\right) + C
\end{aligned}
$$

leading to

$$k\lambda_{Eulerian} = 2\pi \tag{15}$$

Therefore the wavelength of the Eulerian streamlines is given by $\lambda_{Eulerian} = 2\pi/k$.

(d) The wavelength of the Lagrangian trajectory is given by the distance traveled by a fluid element in the x_1-direction between two consecutive crests.

The period of time T to travel between two consecutive crests satisfies: $x_2(t) = x_2(t+T)$ such that

$$\begin{aligned}
x_2(t) &= \frac{V_0}{k(U_0-c)} sin\left[k((U_0-c)t + X_1)\right] + X_2 - \frac{V_0}{k(U_0-c)} sin(kX_1) = \\
x_2(t+T) &= \frac{V_0}{k(U_0-c)} sin\left[k((U_0-c)\cdot(t+T) + X_1)\right] + X_2 - \frac{V_0}{k(U_0-c)} sin(kX_1) \\
x_2(t+T) &= \frac{V_0}{k(U_0-c)} sin\left[k(U_0-c)t + k(U_0-c)T + kX_1\right] + X_2 - \frac{V_0}{k(U_0-c)} sin(kX_1)
\end{aligned}$$

Therefore the period T is given by

$$T = \frac{2\pi}{k(U_0-c)} \tag{16}$$

The Lagrangian trajectory by definition will be the velocity of a fluid element in the x_1-direction times the period T:

$$\lambda_{Lagrangian} = U_0 T = \frac{2\pi U_0}{k(U_0-c)} \tag{17}$$

or

$$\lambda_{Lagrangian} = \frac{U_0}{U_0-c}\lambda_{Eulerian} \tag{18}$$

(e) For $U_0 - c << U_0$, we have $\frac{U_0}{U_0-c} >> 1$ such that $\lambda_{Lagrangian} >> \lambda_{Eulerian}$. For this particular case where $U_0 - c << U_0$ (equivalent to $U_0 \approx c$), the fluid parcels oscillate very slowly compared to the streamlines of the flow. You can compare the plots above that I've draw for this specific case. The Matlab code for the plots is on the course homepage.