EPS 131 Problem set 7 SOLUTIONS

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1. We found in class that gravity waves satisfy

$$\varphi = \frac{a\omega}{k}\frac{\cosh k(z+H)}{\sinh kH}\sin(kx-\omega t)$$

where φ is the "velocity potential", $u=\frac{\partial\varphi}{\partial x}$, $w=\frac{\partial\varphi}{\partial z}$

We're interested in the limit $kH \gg 1$. Expanding terms in the relevant part of φ, we can write

$$f \equiv \frac{\cosh k(z+H)}{\sinh kH} = \frac{e^{kH(1+z/H)}+e^{-kH(1+z/H)}}{e^{kH}+e^{-kH}} \simeq \frac{e^{kH(1+z/H)}}{e^{kH}} = e^{kz}$$

where I have taken the $kH \gg 1$ limit holding the fractional depth $\left(\frac{z}{H}\right)$ constant; the approximation holds anywhere above the bottom $(z=-H)$.

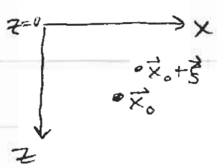
In this regime,

$$\boxed{\varphi \simeq \frac{a\omega}{k}e^{kz}\sin(kx-\omega t)}$$

So the velocities are

$$(u,w) = \left(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial z}\right) = a\omega e^{kz}\left(\cos(kx-\omega t), \sin(kx-\omega t)\right)$$

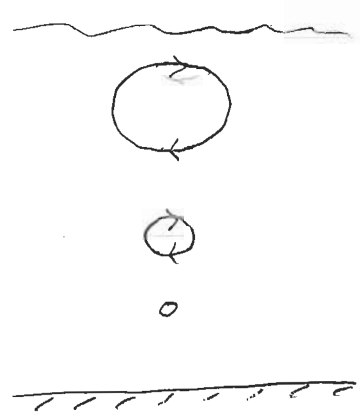
Consider a small amplitude trajectory $\vec{\xi}$ around $\vec{x}_0$ $(k|\vec{\xi}| \ll 1)$.



We can assume the velocity field is fairly constant along this trajectory, so

$$\boxed{\vec{\xi} = \int \vec{u}(\vec{x}_0)\cdot dt = a e^{kz_0}\left(-\sin(kx_0-\omega t), \cos(kx_0-\omega t)\right)}$$

Which is a circular trajectory with radius $ae^{kz_0} = ae^{-k|z_0|}$ (since $z_0<0$) decreasing exponentially with depth.



2\. a) Relaxing the $kH \gg 1$ solution and again considering small trajectories $\vec{\xi} = (\xi, \zeta)$, $k|\vec{\xi}| \ll 1$, we can write

$$\vec{\xi} = \int \vec{u}(\vec{x}_0)\, dt = \int \nabla \varphi(\vec{x}_0)\, dt, \quad \text{giving}$$

$$\xi = -a \frac{\cosh k(z_0+H)}{\sinh kH} \sin(kx_0 - \omega t)$$

$$\zeta = a \frac{\sinh k(z_0+H)}{\sinh kH} \cos(kx_0 - \omega t)$$

Solving the first equation for $\sin(kx_0 - \omega t)$ and the second for $\cos(kx_0 - \omega t)$ allows us to write

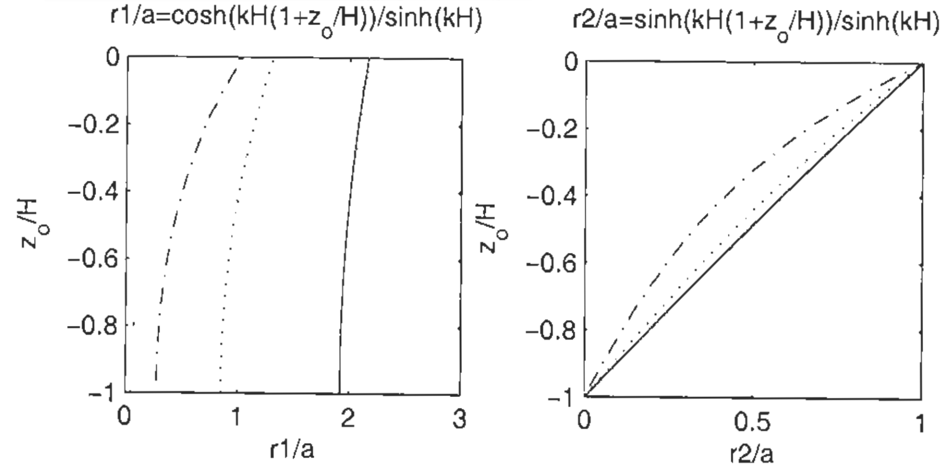
$$1 = \cos^2(kx_0 - \omega t) + \sin^2(kx_0 - \omega t) = \frac{\xi^2}{r_1^2} + \frac{\zeta^2}{r_2^2} \quad \text{with}$$

$$r_1 = a \frac{\cosh k(z_0+H)}{\sinh kH}, \quad r_2 = a \frac{\sinh k(z_0+H)}{\sinh kH}$$

This is the equation for an ellipse, $\left(\frac{x}{r_1}\right)^2 + \left(\frac{y}{r_2}\right)^2 = 1$.

b) The two axes of the ellipse are r_1 and r_2, defined above.

c) Below r_1 and r_2 are plotted for several ratios of the horizontal wavelength to the depth ($\sim kH$).



d) Near the bottom, $r_2 \to 0$ while $r_1 \neq 0$, so the trajectories become horizontal oscillations. $\longleftrightarrow$

3\. As shown in class, when $H \gg \lambda$ ($kH \gg 1$), $\omega = \sqrt{gk}$, and

$$C_g = \frac{\partial \omega}{\partial k} = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{1}{2} C_p \quad \left[C_p = \frac{\omega}{k}\right],$$

i.e., the group speed (speed of packet) is half the phase speed (speed of crests).