EPS 131 Problem set #2 SOLUTION

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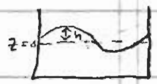
1. Momentum equ'n: $\rho\left(\frac{d\vec{u}}{dt} + 2\vec{\Omega}\times\vec{u}\right) = -\nabla p - \vec{g}\rho + f_r$

vert mom equ'n: $\rho\left(\underbrace{\frac{\partial w}{\partial t}}_{\text{steady}} + \underbrace{\vec{u}\cdot\nabla w}_{\text{small}} - \underbrace{2\Omega u\cos\theta}_{\text{small}}\right) = -\frac{\partial p}{\partial z} - g\rho + \underbrace{f_r}_{\text{small}}$

$$\frac{\partial p}{\partial z} = -\rho g \quad \text{("hydrostatic approximation")}$$



$$\int_z^h dz' \left\{\frac{\partial p}{\partial z} = -\rho g\right\} \Rightarrow \underbrace{p(h)}_{1\,\text{atm} = \text{small}} - p(z) = -\rho_0 g(h-z) \quad \text{assuming } \rho=\rho_0=\text{const}$$

$$\boxed{p(z) = \rho_0 g\left(h(x,y) - z\right)} \quad \text{(pressure is weight of column above)}$$

Gulf stream: $h \simeq h(x)$

$$v = \frac{1}{f\rho_0}\frac{\partial p}{\partial x} = \frac{g}{f}\frac{\partial h}{\partial x} \simeq \frac{g}{f}\frac{\Delta h}{\Delta x}$$

from attached figure, $\Delta h \simeq 1\,\text{m}$, $\Delta x \simeq 0.1° \simeq 100\,\text{km}$

$$f = 2\Omega\sin\theta = 2\cdot\frac{2\pi}{1\,\text{day}}\cdot\sin 30° = 7\cdot10^{-5}\ \text{s}^{-1}$$

$$\boxed{v \simeq 1.4\ \text{m/s}}$$

$$\text{transport} = H\Delta x\, v = (2\,\text{km})(100\,\text{km})(1.4\,\text{m/s}) = 2.8\cdot10^{8}\ \text{m}^3/\text{s} = \boxed{280\ \text{Sv}}$$

Circumpolar current: $h \simeq h(y)$

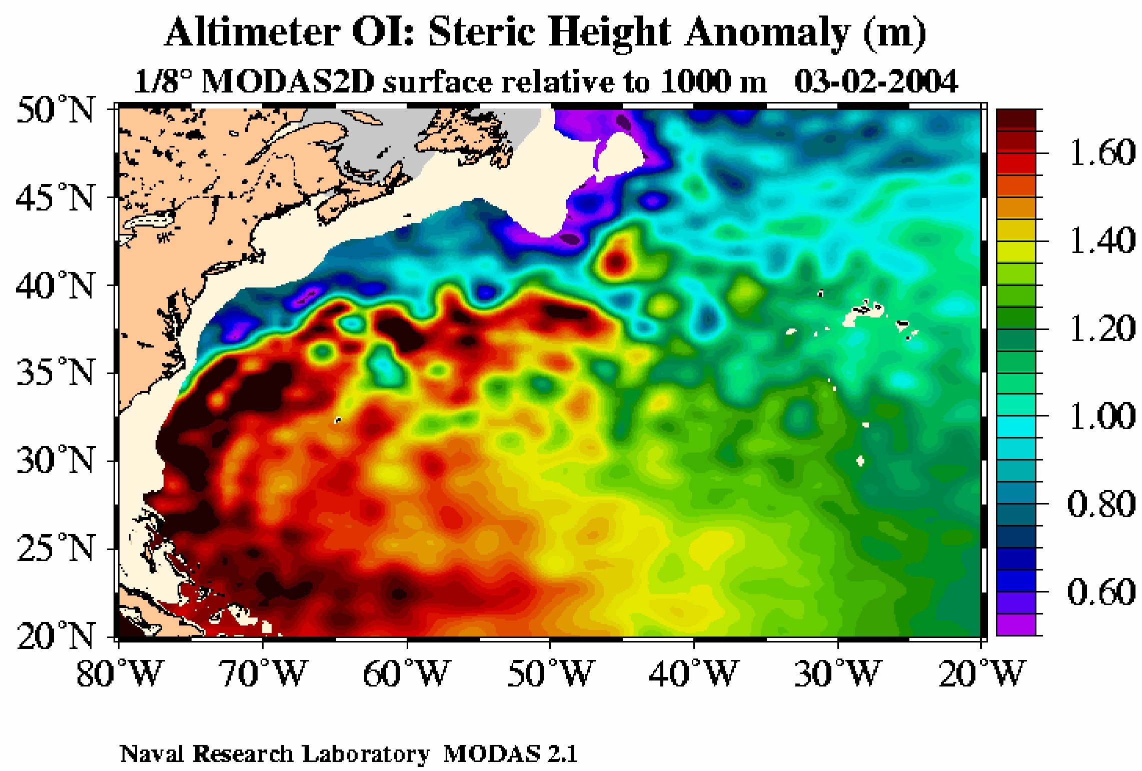
$$u = -\frac{g}{f}\frac{\partial h}{\partial y} \simeq -\frac{g}{f}\frac{\Delta h}{\Delta y} = -\frac{\Delta\psi_g}{\Delta y}\ ; \quad \psi_g \equiv \frac{g}{f}h$$

from attached figure, $\Delta\psi_g \simeq 4.8\cdot10^{4}\ \frac{\text{m}^2}{\text{s}}$, $\Delta y \simeq 700\,\text{km}$; $f = 2\Omega\sin 60° = 1.2\cdot10^{-4}\,\text{s}^{-1}$

$$u \simeq \boxed{6.9\ \text{cm/s}}$$

$$\text{transport} = H\Delta x\, v = 5\,\text{km}\cdot700\,\text{km}\cdot6.9\cdot10^{-2}\,\text{m/s} = \boxed{240\ \text{Sv}}$$

These current estimates for the Gulf stream and ACC are in reasonably good agreement with observations, but the transport estimates are quite high, probably because our assumption that the velocity was constant in height is a bit too liberal. More reasonable values are 120 Sv for the ACC in Drake Passage and 80 Sv for the Gulf stream near Cape Hatteras.



http://www7300.nrlssc.navy.mil/altimetry/images/modas_images/today/gst_ssh.gif

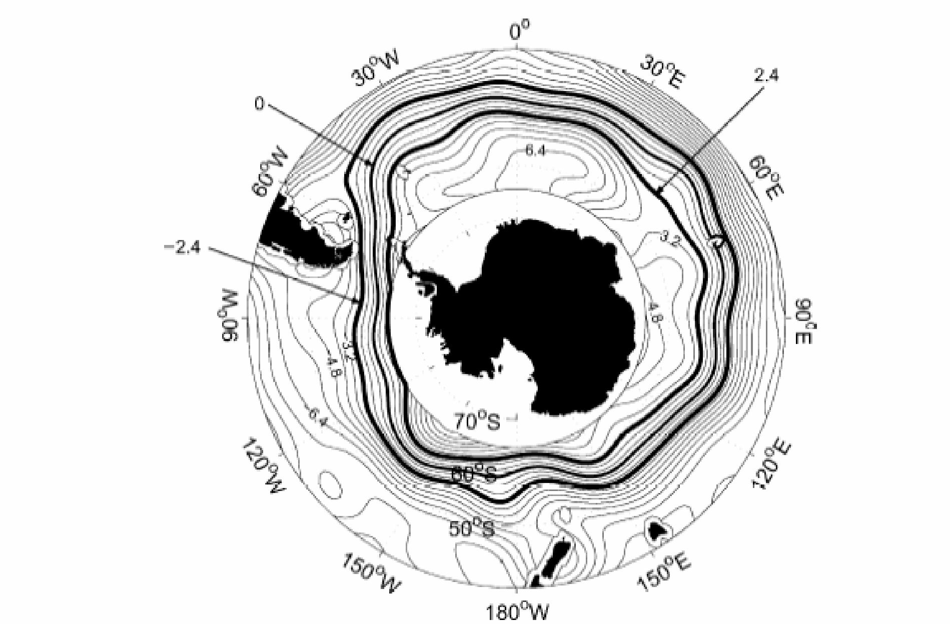


Fig. 1. The geostrophic streamlines, $\Psi_g = gh/f$, with g the gravitational acceleration, h the sea surface height from altimetry and f the Coriolis parameter. The contour interval is $0.8 \times 10^4\ m^2\,s^{-1}$ with the value increasing traveling poleward. The zero contour is located at the axis of the ACC. The bold solid lines mark the boundaries of circumpolar flow: $\Psi_g = \pm 2.4 \times 10^4\ m^2\,s^{-1}$.

http://ace.acadiau.ca/math/Karsten/papers/henotepp.pdf

2. The original problem was somewhat ill posed. If a fluid has trajectories $x(t) = ct^2$, what variable varies from one trajectory to the next? The problem was corrected (see newest version of problem set) to be $x = x_0 e^{ct}$.

i (Lagrangian). $u(x_0,t) = \frac{\partial}{\partial t} x(x_0,t) = c x_0 e^{ct}$

$$a(x_0,t) = \frac{\partial}{\partial t} u(x_0,t) = c^2 x_0 e^{ct}$$

ii (Eulerian u). $x(x_0,t) = x_0 e^{ct} \Rightarrow x_0(x,t) = x e^{-ct}$

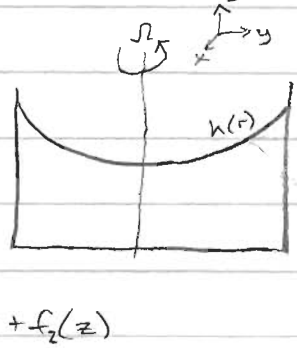
$$u(x_0,t) = c x_0 e^{ct} \Rightarrow \boxed{u(x,t) = cx} \quad \text{steady velocity field}$$

iii (Eulerian a = fluid parcel's acceleration in Eulerian coordinates, $a(x,t)$; not "local acceleration" $= \frac{\partial u(x,t)}{\partial t}$ in Knauss, p. 82)

$$a(x,t) = \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right) u(x,t) = \cancelto{0}{\frac{\partial (cx)}{\partial t}} + cx\,\frac{\partial (cx)}{\partial x}$$

$$\boxed{a(x,t) = c^2 x}$$ Note that $x = x_0 e^{ct}$ shows this agrees w/ i.

3. The centripetal term $\vec{\Omega} \times \vec{\Omega} \times \vec{x}$ effectively changes the gravitational force every particle feels and changes the shape of the earth from a sphere ○ to an ellipsoid ⬭. The change in shape is pretty small and has minimal effect on the dynamics of ocean flow. Since the coriolis term, $2\vec{\Omega} \times \vec{u}$, depends on velocity, it has a far more significant influence on the dynamics.



4. $\rho\left(\frac{d\vec{u}}{dt}\right)_{fixed} = \rho\left(\left(\frac{d\vec{u}}{dt}\right)_{rot} + 2\vec{\Omega}\times\vec{u} + \vec{\Omega}\times\vec{\Omega}\times\vec{x}\right) = -\nabla p - \rho g \hat{z}$

In rotating frame, $\vec{u} = 0$

$$\nabla p = -\rho g \hat{z} - \rho\,\vec{\Omega}\times\vec{\Omega}\times\vec{x} = -\rho g \hat{z} + \rho\Omega^2 r\,\hat{r}$$

$$\frac{\partial p}{\partial z} = -\rho g \Rightarrow p = -\rho g z + f_1(r) \; ; \quad \frac{\partial p}{\partial r} = \rho\Omega^2 r \Rightarrow p = \rho\frac{\Omega^2}{2} r^2 + f_2(z)$$

$\hookrightarrow p(z,r) = -\rho g z + \rho\frac{\Omega^2}{2} r^2 + \underbrace{\phi}_{\text{constant}}$. At surface, $p(h(r),r) = 0 = -\rho g h(r) + \rho\frac{\Omega^2}{2}r^2 + \phi$

$$\boxed{h(r) = h_0 + \frac{\Omega^2}{2g} r^2} \quad \left[h_0 \equiv \frac{\phi}{\rho g}\right]$$