AM203 HOMEWORK #7 SOLUTION

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① The crude way to do this is to iterate the map initially for ~1000 iterations until initial transients decay and then say it's a 2^n-cycle if 2^n additional iterations change x by less than some small tolerance. Varying r and applying this method, the values r_n where a new 2^n-cycle is born can be approximated. Strogatz calls this a "naive approach" (problem 10.6.1) because near the period doubling the convergence to a cycle becomes very slow ("critical slowing down").

At any rate, a method like this leads to the following values:

$x_{n+1} = r\sin(\pi x_n)$: $r_1 \simeq 0.713$, $r_2 \simeq 0.831$, $r_3 \simeq 0.858$

$\hookrightarrow \delta \simeq \frac{r_2 - r_1}{r_3 - r_2} = \boxed{4.4}$

$x_{n+1} = r - x_n^4$: $r_1 \simeq 0.746$, $r_2 \simeq 1.113$, $r_3 \simeq 1.161 \Rightarrow \delta \simeq \boxed{7.6}$

The sine map is unimodal, and δ is reasonably close to the $n \to \infty$ value of $\delta \simeq 4.669$

The quartic map is not unimodal (smooth, concave down) since it's not concave at the origin. Briggs (1991) gives $\delta = 7.28$ for quartic maps.

(2) (Note that α is positive in Schuster, and in this problem, but negative in Strogatz)

a) Given $g(x) = -\alpha g(g(-\frac{x}{\alpha})) \equiv T(g)$

Define $f(x) \equiv \mu g(\frac{x}{\mu})$

$T(f) = -\alpha f(f(-\frac{x}{\alpha})) = -\alpha\mu g(\frac{1}{\mu}\mu g(-\frac{1}{\mu}\frac{x}{\alpha}))$

$= -\alpha\mu g(g(-\frac{x}{\mu\alpha}))$

Since $g(x) = -\alpha g(g(-\frac{x}{\alpha}))$, $g(\frac{x}{\mu}) = -\alpha g(g(-\frac{x}{\mu\alpha}))$, so

$T(f) = \mu g(\frac{x}{\mu}) = f$

$\hookrightarrow$ $f \equiv \mu g(\frac{x}{\mu})$ is a fixed point of T.

b) If x^* is a f.p., $g(x^*) = x^*$, then

$x^* = g(x^*) = -\alpha g(g(-\frac{x^*}{\alpha}))$. Similarly,

$g(-\alpha x^*) = -\alpha g(g(x^*)) = -\alpha g(x^*) = -\alpha x^*$

But $g(-\alpha x^*) = g(\alpha x^*)$. So $g(z) = \pm z$ at all $\pm\alpha^n x^*$

c) Let $g(x) = 1 + cx^2 + \mathcal{O}(x^3)$. Since $g = T(g)$,

$1 + cx^2 \simeq -\alpha\left[1 + c\left(1 + 2c\frac{x^2}{\alpha^2} + \mathcal{O}(x^4)\right)\right] \quad \forall x$

So, $1 = -\alpha(1+c)$ and $c = -\alpha\frac{2c^2}{\alpha^2}$

$\hookrightarrow$ $c = -\frac{1}{2}(1+\sqrt{3}) \simeq -1.4$; $\alpha = 1+\sqrt{3} \simeq 2.7$

Which is surprisingly close to the exact value, $\alpha = 2.5029\ldots$

(3) Following Schuster (p. 46),

$$g_i(x) = \lim_{n\to\infty} (-\alpha)^n f^{2^n}_{R_{n+i}}\left(\frac{x}{(-\alpha^n)}\right)$$

$$g_{i-1}(x) = \lim_{n\to\infty} (-\alpha)^n f^{2^n}_{R_{n+i-1}}\left(\frac{x}{(-\alpha)^n}\right) = \lim_{n\to\infty} (-\alpha)(-\alpha)^{n-1} f^{2^{n-1+1}}_{R_{n-1+i}}\left(-\frac{1}{\alpha}\frac{x}{(-\alpha)^{n-1}}\right)$$

define $m \equiv n-1$

$$g_{i-1}(x) = \lim_{m\to\infty} (-\alpha)(-\alpha)^m f^{2^{m+1}}_{R_{m+i}}\left(-\frac{1}{\alpha}\frac{x}{(-\alpha)^m}\right)$$

$$= \lim_{m\to\infty} (-\alpha)(-\alpha)^m f^{2^m}_{R_{m+i}}\left[\frac{1}{(-\alpha)^m}(-\alpha)^m f^{2^m}_{R_{m+i}}\left(\frac{(-x/\alpha)}{(-\alpha)^m}\right)\right]$$

$$= -\alpha g_i\left[g_i\left(-\frac{x}{\alpha}\right)\right]$$

$\hookrightarrow g_{i-1}(x) = T[g_i(x)]$

(4) a) This is directly from Strogatz. Please see me if you have questions.

b [the easier part]) If $x_{n+1} = f(x_n) = r - x_n^4$, $g(x)$ be quartic to lowest order:

$$g(x) \doteq 1 + cx^4 + O(x^8) = -\alpha g^2\left(-\frac{x}{\alpha}\right) = -\alpha\left[1 + c\left(1 + 4c\left(\frac{x}{\alpha}\right)^4 + O(x^8)\right)\right]$$

$$1 = -\alpha(1+c) \quad \text{and} \quad c = -4\alpha\frac{c^2}{\alpha^4}$$

$\hookrightarrow c = -1.54, \quad \alpha = 1.84$

Briggs (1991) gives $\alpha = 1.69$ for quartic maps.