Homework #5

Nonlinear dynamics and chaos

1. Estimate the period of the limit cycle of the following system for $k \gg 1$:

$$\ddot{x} + k(x^2 - 4)\dot{x} + x = 1$$

2. Consider the equation

$$\ddot{x} + \varepsilon \dot{x}^3 + x = 0$$

   (a) Derive the averaged equations.

   (b) Given the initial conditions $x(0) = a$, $\dot{x}(0) = 0$, solve the averaged equations and thereby find an approximate formula for $x(t, \varepsilon)$.

   (c) (**This section is optional and extra credit:**) Solve the equation numerically in the range $0 \leq t \leq 50$. (If unfamiliar with solving ODEs in Matlab, you can look at lorenz2.m on the course homepage for an example.) Plot the results of the numerical solution next to the approximate answer from part (b): using $a = 1$, see how well the two solutions agree for $\varepsilon = 0.1, 0.5, 1, 2, 5, 10$. Notice the impressive aggreement, even when ε is not small! How (qualitatively) does the agreement between the numerical and analytical approximations at $\varepsilon = 2$ depend on the value of a?

3. By plotting phase portraits with Matlab, show that the system

$$\begin{aligned} \dot{x} &= -y + \mu x + x y^2 \\ \dot{y} &= x + \mu y - x^2 \end{aligned}$$

   undergoes a Hopf bifurcation at $\mu = 0$. Is it subcritical, supercritical, something else?

4. (A tough one, it's the effort that counts here…) Consider the system

$$\begin{aligned} \ddot{u} + \omega^2 u &= (\varepsilon - \alpha z)\dot{u} \\ \dot{z} + \tau z &= u^2 \end{aligned}$$

   ω and α are constants, τ is a positive constant that is away from zero, and ε is a small positive parameter.

(a) Use the method of multiple scales and show that, to the first approximation,

$$u \approx a\cos(\omega t + \beta)$$

where

$$\begin{aligned} \dot{a} &= \frac{1}{2}\varepsilon a - \frac{\alpha(\tau^2 + 8\omega^2)}{8\tau(\tau^2 + 4\omega^2)}a^3 \\ \dot{\beta} &= -\frac{\alpha\omega}{4(\tau^2 + 4\omega^2)}a^2. \end{aligned}$$

(b) Which bifurcation do these equations describe?

(c) As function of which parameter?

(d) Why does the multiple scale approximation make sense to use here?