AM 203 Homework 5 SOLUTIONS

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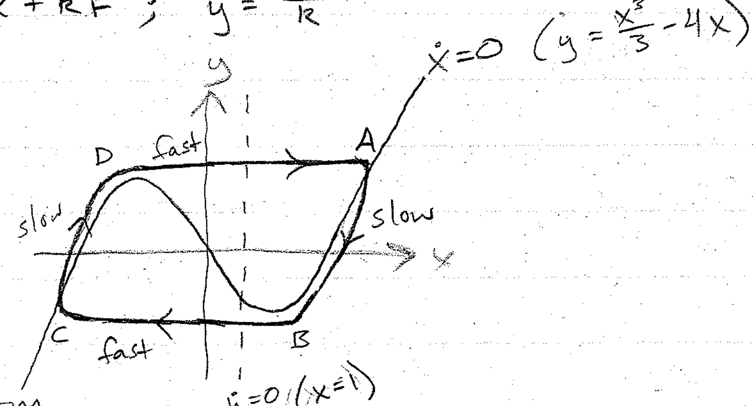
(1) $\ddot{x} + k(x^2-4)\dot{x} + x = 1, \quad k \gg 1$

$\ddot{x} + k\dot{x}(x^2-4) = \frac{d}{dt}\left[\dot{x} + k\left(\tfrac{1}{3}x^3 - 4x\right)\right] = 1 - x$

Let $F \equiv \frac{1}{3}x^3 - 4x$; $\omega \equiv \dot{x} + kF$; $y \equiv \frac{\omega}{k}$

$\dot{x} = k\left(y - F(x)\right)$

$\dot{y} = -\frac{1}{k}(x-1)$



When $y \simeq F(x)$, $\dot{x} \simeq 0$ and $\dot{y} \sim \frac{1}{k} \Rightarrow$ slow branches.

Otherwise, $\dot{y} \sim \frac{1}{k}$ is slow and $\dot{x} \sim k$ causes rapid motion toward $y = F$. Note that the trajectory is roughly symmetric about $y=0$: the offset of the nullcline causes slower motion between $A \to B$ than $C \to D$.

Trajectory spends almost all time on slow branches: Period $\simeq T_{AB} + T_{CD}$

$T_{AB} = \int_{x_A}^{x_B} \frac{dx}{\dot{x}}$; $y \simeq F(x)$ so $\dot{y} = F'(x)\dot{x} \Rightarrow \dot{x} = \frac{\dot{y}}{F'(x)} = \frac{1-x}{k(x^2-4)}$

$T_{AB} = k\int_{x_A}^{x_B} \frac{x^2-4}{1-x}\,dx$

$T_{CD} = k\int_{x_C}^{x_D} \frac{x^2-4}{1-x}$

Find x_A, x_B, x_C, x_D: $(x_D, x_B) = \text{extrema}\left(\tfrac{1}{3}x^3 - 4x\right) \Rightarrow x_B = 2,\ x_D = -2$

$F(x_A) = F(x_D) = F(-2) = \frac{16}{3} = \frac{1}{3}x_A^3 - 4x \Rightarrow x_A = 4,\ x_C = -x_A = -4$

$\tau = T_{AB} + T_{CD} = k\left\{\int_4^2 \frac{x^2-4}{1-x}\,dx + \int_{-4}^{-2} \frac{x^2-4}{1-x}\,dx\right\} = k\left\{(8 - 3\ln 3) + (4 - 3\ln\tfrac{5}{3})\right\}$

[Evaluate integral, e.g., with $x' = 1-x$ and expand integrand]

$\boxed{\tau = 3k(4 - \ln 5)} \simeq 7.2k$

(2) $\ddot{x} + \varepsilon \dot{x}^3 + x = 0$

a) Following Strogatz p.223-224, $x_0 = r(T)\cos\theta$; $\theta \equiv \tau + \phi(T)$

$r' = \langle h \sin\theta \rangle$

$r\phi' = \langle h \cos\theta \rangle$

Here $h(x,\dot{x}) = h(r\cos\theta, -r\sin\theta) = \dot{x}^3 = -r^3 \sin^3\theta$

$r' = \langle -r^3 \sin^4\theta \rangle = -\frac{3}{8} r^3$ (using $\langle \sin^4\theta \rangle = \frac{3}{8}$)

$r\phi' = \langle -r^3 \sin^3\theta \cos\theta \rangle = 0$

$\hookrightarrow \boxed{r' = -\frac{3}{8} r^3, \ \phi' = 0}$

b) Initial condition: Consider $x(0)$ and $\dot{x}(0)$ given (independent of ε). Then we have $x_1(0) = \dot{x}_1(0) = 0$ and $x(0) = x_0(0)$, $\dot{x}(0) = x_0(0)$. But $\dot{x}(0) = (\partial_\tau + \varepsilon \partial_T) x_0 = -r\sin\theta + O(\varepsilon)$

Now we have $\sqrt{x(0)^2 + \dot{x}(0)^2} = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + O(\varepsilon)} \simeq r$.

Similarly $\phi(0) \simeq \tan^{-1}\left(\frac{\dot{x}(0)}{x(0)}\right)$ [cf Strogatz p. 225]

Here, $x(0) = a$, $\dot{x}(0) = 0 \Rightarrow \boxed{r(0) \simeq a, \ \phi(0) \simeq 0}$

$\frac{dr}{dT} = -\frac{3}{8} r^3 \Rightarrow \int r^{-3} dr = -\frac{3}{8} \int dT \Rightarrow r^{-2} = \frac{3}{4} T + c$

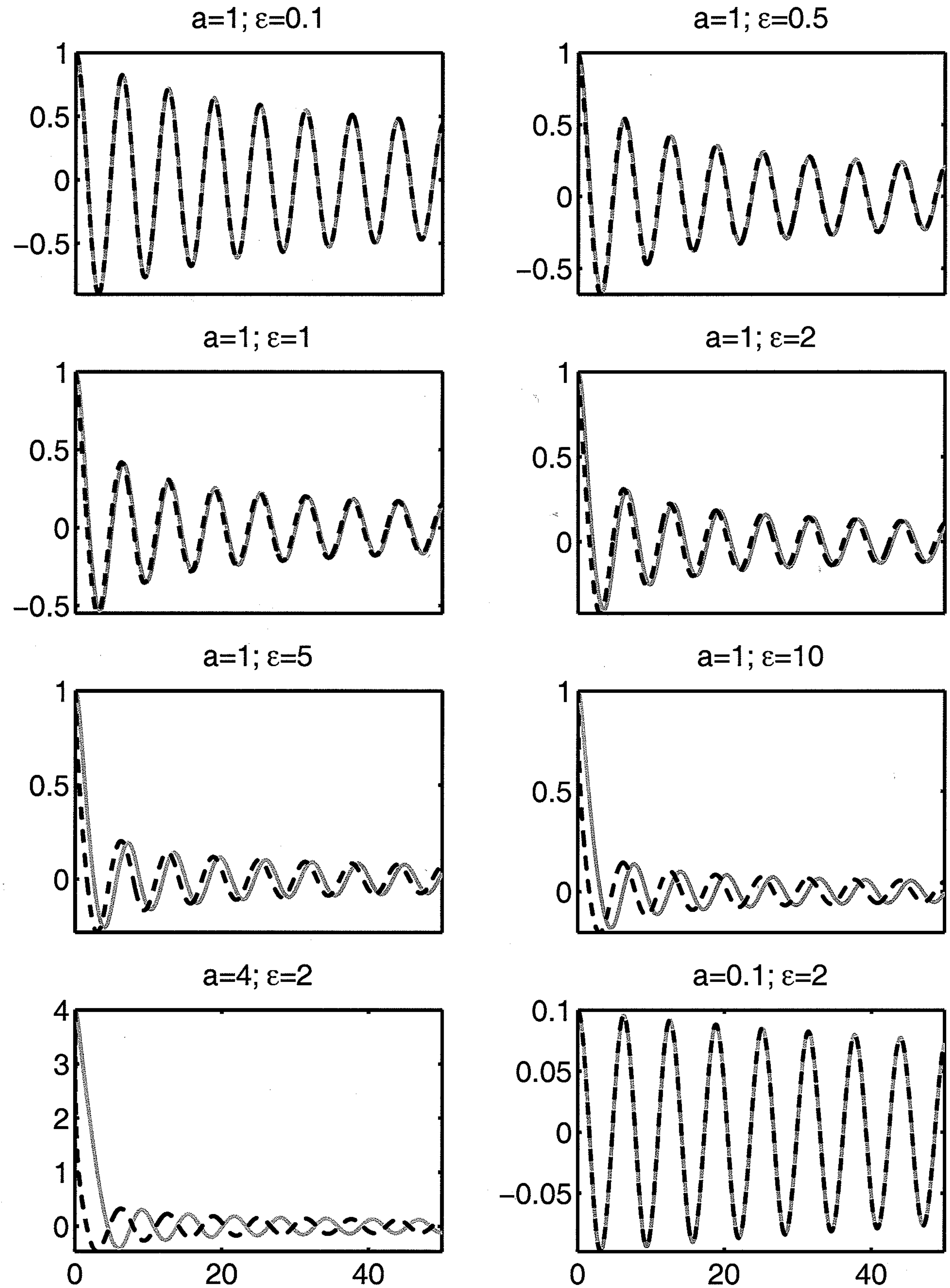
at $T = 0$, $r = c^{-1/2} = a \Rightarrow c = a^{-2}$

$r(T) = \left(\frac{3}{4} T + a^{-2}\right)^{-1/2} = \frac{2}{\sqrt{4/a^2 + 3T}}$

Since $\phi' = \phi(0) = 0$, $\phi(T) = 0$

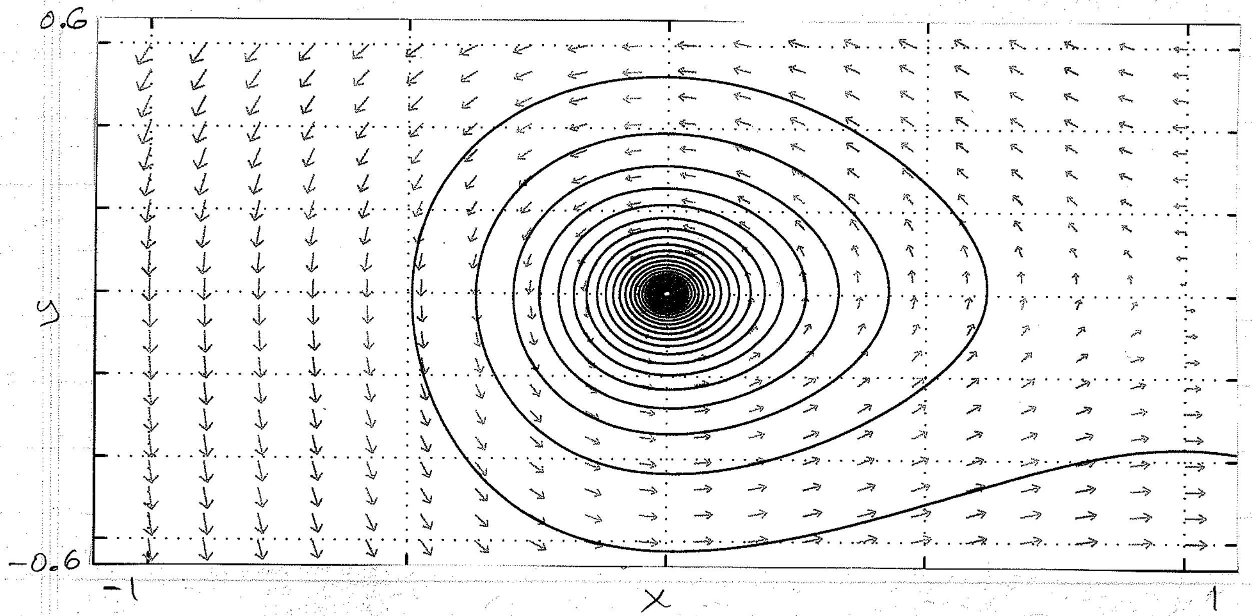
$$\boxed{x(t) = \frac{2}{\sqrt{4/a^2 + 3\varepsilon t}} \cos t + O(\varepsilon)}$$

c) On the next page, x vs. t is plotted for various values of ε, a; analytical approximation is black dash, numerical is gray line. Amplitude initially dies faster for larger ε, A, so time scale separation is less accurate approximation in these regions of (ε, A, t) space.

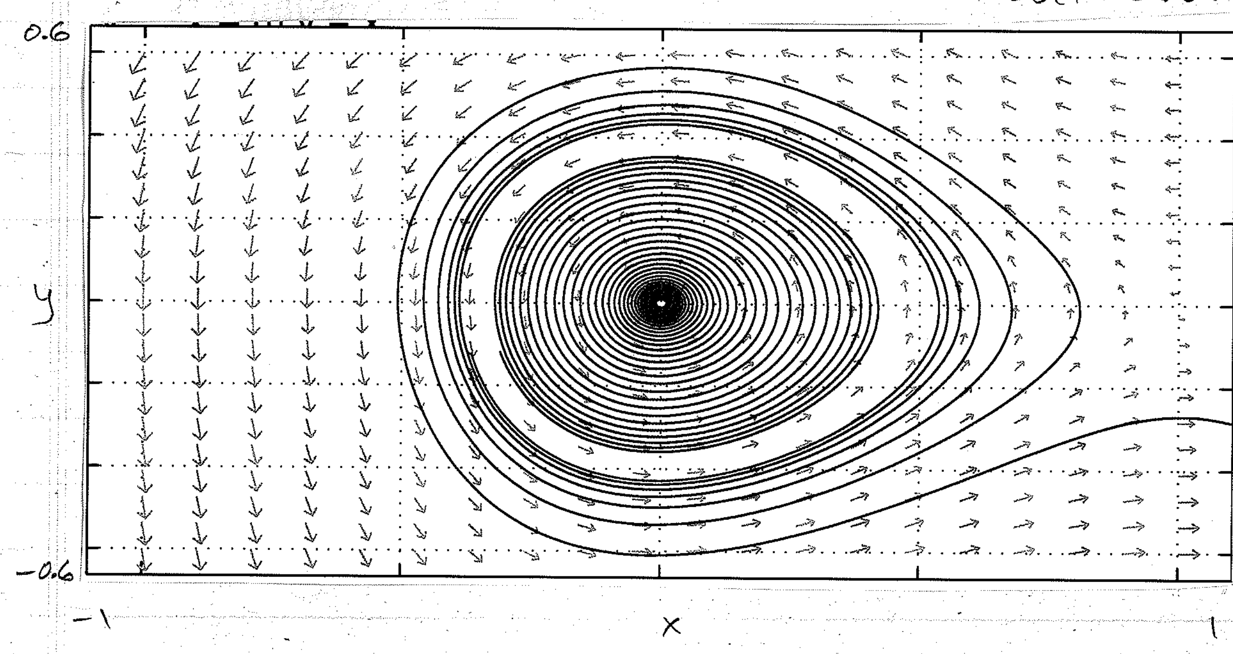

a=1; ε=0.1
a=1; ε=0.5
a=1; ε=1
a=1; ε=2
a=1; ε=5
a=1; ε=10
a=4; ε=2
a=0.1; ε=2
1
0.5
0
−0.5
4
3
2
1
0
0.1
0.05
−0.05
20
40

③ When $\mu > 0$, the origin is an unstable f.p.
When $\mu < 0$, origin becomes stable and small unstable limit cycle is born:
<u>subcritical Hopf</u> bifurcation.
I used pplane, plotting only forward trajectories.

$\mu = 0.02$



$\mu = -0.02$



(4) $\ddot{u} + \omega^2 u = (\varepsilon - \alpha z)\dot{u}$

$\dot{z} + \tau z = u^2$ $\qquad \tau > 0, \ \alpha > 0, \ \varepsilon \geq 0, \ \varepsilon \ll 1$

a) Look for limit cycle of form $u = a\cos(\omega t + \beta)$

<u>APPROACH 1</u> (three time scales)

$\dot{y} = (\varepsilon - \alpha z) y - \omega^2 u$

$\dot{u} = y$

$\dot{z} = u^2 - \tau z$

Origin is fixed point. With $\varepsilon = 0$, it is stable. We're looking for a limit cycle that might emerge for $\varepsilon > 0$. Let the amplitude be $\mathcal{O}(\eta)$ (η undetermined), so $u = \mathcal{O}(\eta)$, $y = \mathcal{O}(\eta)$, $z = \mathcal{O}(\eta^2)$. Since ε shows up as $(\varepsilon - \alpha z)$, ε becomes important at $\mathcal{O}(\eta^2)$. So we're interested in $\varepsilon \sim \mathcal{O}(\eta^2)$, and we can write $\varepsilon = E\eta^2$ with $E = \mathcal{O}(1)$.

Expand in η, introducing slow time scales T_n:

$\frac{d}{dt} = \partial_{T_0} + \eta\,\partial_{T_1} + \eta^2\,\partial_{T_2} + \dots$

$(u, y, z) = \sum_{n=1}^{3} \eta^n \left(u_n(T_0, T_1, T_2), y_n, z_n\right)$ $\qquad$ [perhaps confusing notation...]

$\varepsilon = \eta^2 E$

Equate terms:

$\mathcal{O}(\eta^1)$: $\partial_{T_0} y_1 = -\omega^2 u_1$

$\partial_{T_0} u_1 = y_1$

$\partial_{T_0} z_1 = -\tau z_1$

$\hookrightarrow$ $\boxed{u_1 = A(T_1, T_2)\, e^{i\omega T_0} + c.c.}$

$\boxed{y_1 = i\omega A e^{i\omega T_0} + c.c.}$

$z_1 = B(T_1, T_2)\, e^{-\tau T_0}$

Since τ, T_0 are $\mathcal{O}(1)$, $z_1 \to 0$ much faster than variations in $A(T_1)$ ($T_1 = \mathcal{O}(\eta)$), so take $\boxed{z_1 \simeq 0}$ (as expected from discussion above).

④ a) cont'd

$\mathcal{O}(\eta^2)$: $\partial_{T_1} y_1 + \partial_{T_0} y_2 = -\omega^2 u$

$\partial_{T_1} u_1 + \partial_{T_0} u_2 = y_2$

$\partial_{T_1} z_1 + \partial_{T_0} z_2 = u_1^2 - \tau z_2$

$\hookrightarrow$ $\partial_{T_0} y_2 + \omega^2 u_2 = -\partial_{T_1} A\, i\omega\, e^{i\omega T_0} + c.c.$

$\partial_{T_0} u_2 - y_2 = -\partial_{T_1} A e^{i\omega T_0} + c.c.$

$\partial_{T_0} z_2 + \tau z_2 = (A e^{i\omega T_0} + c.c.)^2 - 0$

Avoid secular terms: $\partial_{T_1} A = 0 \Rightarrow A = A(T_2) \Rightarrow \boxed{u_2 = y_2 = 0}$

Integrate z_2 equ'n: $\boxed{z_2 = \frac{2|A|^2}{\tau} + \frac{A^2}{2i\omega + \tau} e^{2i\omega T_0} + c.c.}$

$\mathcal{O}(\eta^3)$: $\partial_{T_2} y_1 + \partial_{T_0} y_3 = E y_1 - \alpha y_1 z_2 - \omega^2 u_3$

$\partial_{T_2} u_1 + \partial_{T_0} u_3 = y_3$

$\partial_{T_1} z_2 + \partial_{T_0} z_3 = -\tau z_3$

Eliminate u_3 between first 2 equations:

$$\partial_{T_0}^2 y_3 + \omega^2 y_3 = \partial_{T_0 T_2} u_1 - \partial_{T_0 T_2} y_1 + E\partial_{T_0} y_1 - \alpha \partial_{T_0} y_1 z_2 - \alpha y_1 \partial_{T_0} z_2$$

After some messy algebra, find that secular terms eliminated if

$$A' = \frac{1}{2} E A - \frac{\alpha}{\tau} A|A|^2 + \frac{\alpha}{2} A \frac{|A|^2}{2i\omega + \tau}, \qquad A' \equiv \partial_{T_2} A$$

We had $u_1 = A e^{i\omega T_0} + c.c.$. Let $A = \frac{1}{2} a e^{i\beta}$

Let $A = \frac{1}{2\eta} a e^{i\beta} \Rightarrow u \simeq \eta u_1 = a\cos(\omega t + \beta)$

$$\hookrightarrow \boxed{\begin{aligned} \dot{a} &= \frac{1}{2}\varepsilon A - \frac{\alpha(\tau^2 + 8\omega^2)}{8\tau(\tau^2 + 4\omega^2)} a^3 \\ \dot{\beta} &= -\frac{\alpha\omega}{4(\tau^2 + 4\omega^2)} a^2 \end{aligned}}$$

(4) a) cont'd

APPROACH 2 (more intuitive, simpler)

$$\ddot{u} + \omega^2 u + (\alpha z - \varepsilon)\dot{u} = 0 \qquad (1)$$

$$\dot{z} = u^2 - \tau z \qquad (2)$$

Examining (1) we see that when $(\alpha z - \varepsilon) > 0$, $\ddot{u} \sim -\dot{u}$ and there is damping. Equ'n (2) shows that z relaxes toward $\frac{u^2}{\tau}$. So if we start with an initial condition far from the origin, u will be damped and decrease and z will decay toward the small $\frac{u^2}{\tau}$. After initial transients have decayed, $z \sim \frac{u^2}{\tau}$, so $\boxed{z = O(u^2)}$; now the damping term is $(\alpha z - \varepsilon) \sim (\frac{\alpha}{\tau}u^2 - \varepsilon)$, so the solution grows for $(\frac{\alpha}{\tau}u^2 - \varepsilon) < 0$ and decays for $(\frac{\alpha}{\tau}u^2 - \varepsilon) > 0$, and hence we must have a limit cycle of amplitude $u \sim (\varepsilon \frac{\tau}{\alpha})^{1/2}$, or $\boxed{u = O(\sqrt{\varepsilon})}$. This means $\boxed{z = O(u^2) = O(\varepsilon)}$. By analogy with the 1D weakly nonlinear oscillator, which this system closely resembles, we can write the solution as

$$\boxed{u(s,T) = a(T)\cos(\omega s + \beta(T))(1 + O(\varepsilon))}$$

with $\boxed{s \equiv t; \quad T \equiv \varepsilon t}$

Based on the scaling arguments above, we can now write the system in term of all $O(1)$ terms except $\varepsilon \ll 1$; introducy $a = \varepsilon^{1/2} A$:

$$u \simeq \varepsilon^{1/2} A(t) \cos\theta + \varepsilon^{3/2} u_1 ; \quad \theta \equiv \omega s + \beta(T) \qquad (3)$$

$$z \simeq \varepsilon z_1 \qquad (4)$$

4 a) cont'd

Start by inserting (3), (4) into (2):

$$\partial_s z_1 = A^2 \cos^2(\omega s + \beta) - \tau z_1$$

This can be solved exactly: I used Mathematica.

$$z_1(\tau) = c_1 \overset{\to 0}{e^{-\tau s}} + A^2 \frac{\tau^2 + 4\omega^2 + \tau^2 \cos 2(\omega s+\beta) + 2\omega\tau \sin 2(\omega s + \beta)}{2(\tau^3 + 4\tau\omega^2)} \qquad (5)$$

After initial transients have died out, expect $c_1 e^{-\tau s}$ term to be negligible, so drop it.

Now plug (3) into (1), using $\partial_t = \partial_s + \varepsilon \partial_T$

Define $S \equiv \sin\theta = \sin(\omega s + \beta(T))$; $C \equiv \cos\theta$

$$\partial_{ss} u_0 + \varepsilon(\partial_{ss} u_I + \partial_{sT} u_0) + \varepsilon(\alpha z_1 - 1)\partial_s u_0 + \omega^2 u_0 + \omega^2 \varepsilon u_I = 0$$

$$-\varepsilon^{1/2}\omega^2 AC + \varepsilon^{3/2}(\partial_{ss} u_1 - 2A'S - 2A\beta' C) - \varepsilon^{3/2}(\alpha z_1 - 1)AS + \varepsilon^{1/2}\omega^2 AC + \varepsilon^{3/2}\omega^2 u_1 = 0$$

[Note that here $u_0 = \varepsilon^{1/2} AC$ and $u_I = \varepsilon^{1/2} u_1$]

$O(\varepsilon^{1/2})$: $-\omega^2 AC + \omega^2 AC = 0$ (by construction)

$O(\varepsilon^{3/2})$: $\partial_{ss} u_1 + \omega^2 u_1 = 2A'S + 2A\beta' C + (\alpha z_1 - 1)AS = \text{RHS}$

To eliminate resonant terms in u_1, we need

$$0 = \int_0^{2\pi/\omega} (\text{RHS}) \sin(\omega s + \beta)\, ds = \int_0^{2\pi/\omega} (\text{RHS}) \sin(\omega s + \beta)\, ds$$

We have z_1 (5). Just need to evaluate integrals and solve first one for A', second one for β'. I used Mathematica for this, getting

$$A' = \frac{1}{2}A - \frac{\alpha(\tau^2 + 8\omega^2)}{8\tau(\tau^2 + 4\omega^2)} A^3, \qquad \beta' = -\frac{\alpha\omega}{4(\tau^2 + 4\omega^2)} A^2$$

(Use $a = \varepsilon^{1/2} A$, $\dot{a} = \varepsilon^{1/2}\dot{A} = \varepsilon^{3/2} A'$, $\beta' = \varepsilon \dot{\beta}$ to get desired result).

Mathematica Notebook $\longrightarrow$

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In[1]:= c = Cos[ω s + B]; S = Sin[ω s + B];

In[2]:= ans = Simplify[DSolve[∂_s z[s] + τ z[s] == A^2 c^2, z[s], s]];

In[3]:= Z1 = z[s] /. ans[[1]] /. C[1] → 0;

In[8]:= R = 2 Ap S + 2 Bp A c - (1 - α Z1) A S;

In[13]:= Simplify[Solve[∫_0^(2π/ω) R S ds == 0, Ap]]

         Simplify[Solve[∫_0^(2π/ω) R c ds == 0, Bp]]
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Out[13]= $\left\{\left\{Ap \to \frac{-A^3 \alpha(\tau^2 + 8\omega^2) + 4A(\tau^3 + 4\tau\omega^2)}{8(\tau^3 + 4\tau\omega^2)}\right\}\right\}$

Out[14]= $\left\{\left\{Bp \to -\frac{A^2 \alpha\omega}{4(\tau^2 + 4\omega^2)}\right\}\right\}$

(4) b) The equation for $\dot{a}$ has roots at $a=0$, $a = \pm 2\sqrt{R}$, $\boxed{R \equiv \varepsilon \frac{\tau(\tau^2+4\omega^2)}{\alpha(\tau^2+8\omega^2)}}$

So we see that an $a>0$ solution exists for $R>0$. Plotting $\dot{a}$ vs. a, or noting that $\dot{a} = \frac{\varepsilon}{2} a - k a^3$, we see that the origin is stable when $R<0$ and it becomes unstable w/ a stable limit cycle appearing when $R>0 \Rightarrow$ <u>supercritical</u> <u>Hopf</u> bifurcation.

c) It occurs as R is varied.

d) There are 3 time scales in this problem: oscillation (t), slowly varying amplitude (εt), and timescale related to coupling w/ z ($\sqrt{\varepsilon}t$). (In Approach 2 I was able to sidestep the third timescale.)