AM 203 Homework #3 SOLUTIONS

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① For any linear system $\dot{\vec{x}} = \underline{A}\vec{x}$ ($\underline{A}$ not singular) there is one fixed point, $\vec{x}^* = \vec{0}$.

a) $\dot{x} = x - y, \quad \dot{y} = x + y \Rightarrow A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$\tau = Tr(A) = 2, \quad \Delta = det(A) = 2 \Rightarrow$ $\boxed{\text{unstable spiral}}$

$\lambda = 1 \pm i, \quad \vec{v} = \begin{pmatrix} \pm i \\ 1 \end{pmatrix}$

Plot shows trajectories spiral CCW

b) $\dot{x} = 5x + 2y; \quad \dot{y} = -17x - 5y \Rightarrow A = \begin{pmatrix} 5 & 2 \\ -17 & -5 \end{pmatrix}$

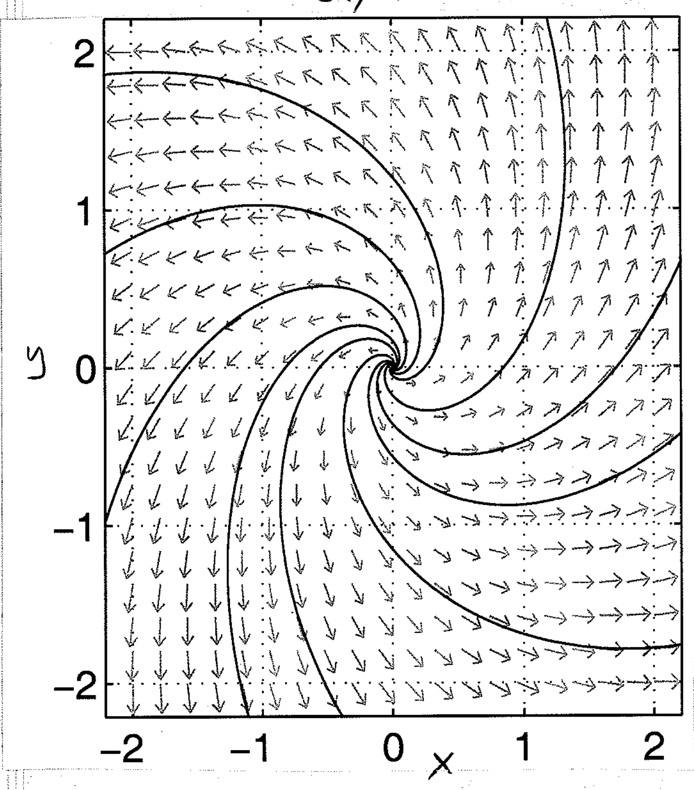
$\tau = Tr(A) = 0, \quad \Delta = det(A) = 9 \Rightarrow$ $\boxed{\text{center}}$

$\lambda = \pm 3i, \quad \vec{v} = \begin{pmatrix} -5 \mp 3i \\ 17 \end{pmatrix}$

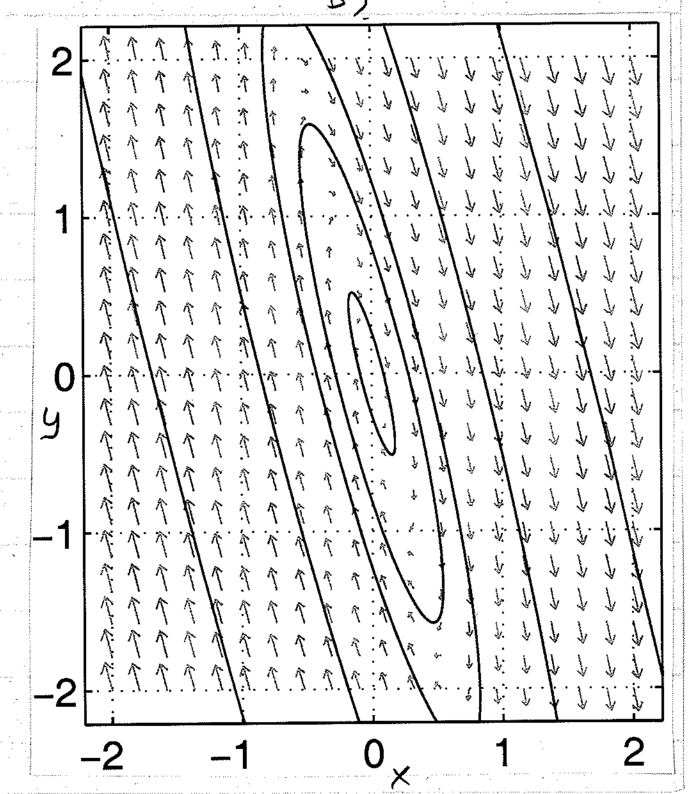
Note that we don't have to worry about stabilizing/destabilizing nonlinear terms even though a center is a "borderline case" because the problem is linear.

Plots generated in matlab with pplane: http://math.rice.edu/dfield

a)



b)

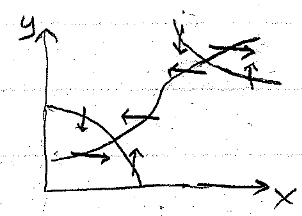


(2) a) $\dot{x} = xy - 1, \quad \dot{y} = x - y^3$

f.p.: $(x^*, y^*) = (\pm 1, \pm 1)$

nullclines: $\dot{x} = 0$ at $x = \frac{1}{y} \quad \left(\dot{y} = \frac{1}{y} - y^3\right)$

$\dot{y} = 0$ at $x = y^3 \quad (\dot{x} = y^4 - 1)$

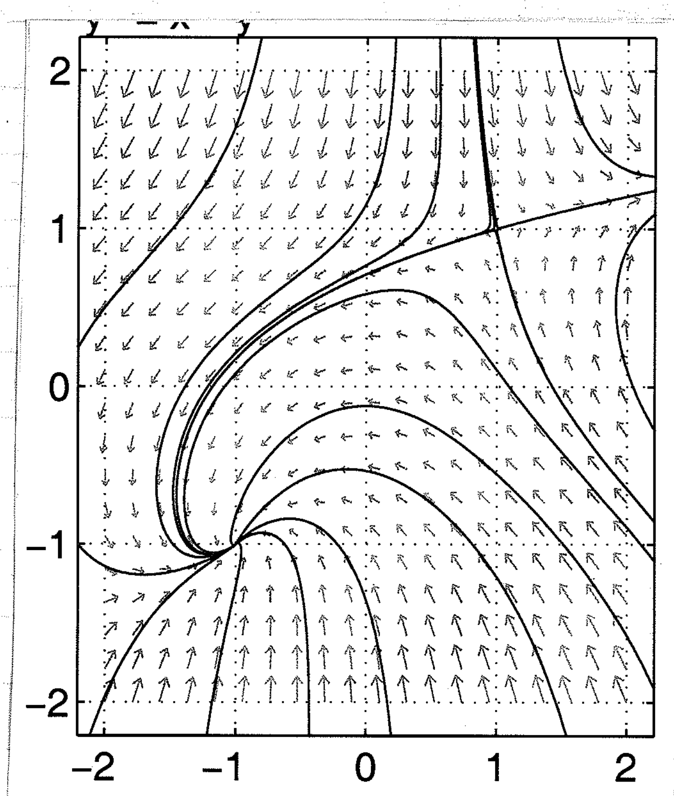


Sketching the vector field along the nullclines, we can guess (1,1) is a <u>saddle</u> and (−1,−1) a <u>stable spiral</u>

Linearize: $J = \begin{pmatrix} y^* & x^* \\ 1 & -3y^{*2} \end{pmatrix} \Rightarrow$ (1,1) has $\tau = -2$, $\Delta = -4$

(−1,−1) has $\tau = -4$, $\Delta = 4$

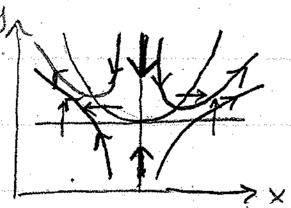
Linearization predicts a saddle at (1,1) and a degenerate node (only 1 e'vector) at (−1,−1) [nonlinear system appears to have stable spiral].



(2) b) $\dot{x} = xy$, $\dot{y} = x^2 - y$

f.p. at $(x^*, y^*) = (0,0)$

nullclines: $\dot{x} = 0$ at $x=0$ $(\dot{y} = -y)$



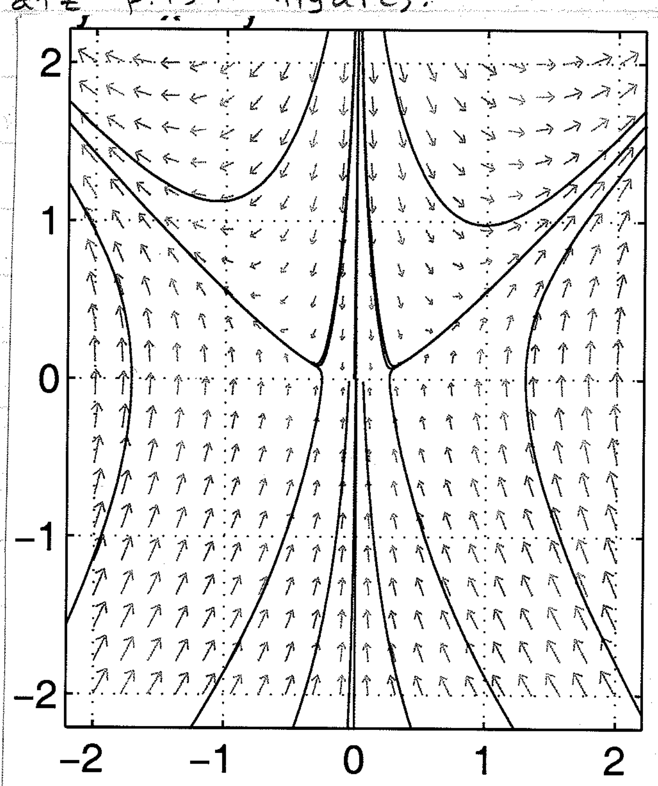
$\dot{x} = 0$ at $y = 0$ $(\dot{y} = x^2)$

$\dot{y} = 0$ at $y = x^2$ $(\dot{x} = x^3)$

The nullclines show that the f.p. is a saddle. Stable manifold is y-axis (one can show the unstable manifold approaches $y = |x|$ far from the origin).

Linearization: $J = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow \tau = -1, \Delta = 0$ $[\lambda = (0,-1)]$

Linear system would have non-isolated fixed points. Nonlinear system clearly doesn't match this "borderline case", because nullclines intersect at a distinct point. Based on linearization, it could then be a stable node or saddle (two adjacent regions in Strogatz p.137 figure).



The x^2 term destabilizes the x-axis, causing there to be no line of stable f.p.s that linearization predicts.

Consider a small perturbation from the origin (f.p.) along the x-axis. It will move up since $\dot{y} \simeq x^2$. Then it will move away from the origin since $\dot{x} = xy$. $\Rightarrow$ instability.

Hence f.p. is a saddle, as shown also w/ nullclines.

③ See attached. For each plot, the i.c. is indicated by Θ_0, along with the values of K, Ω. The winding number w, computed by iterating many times ($N=1000$) without mod 1 and using $w = \frac{\Theta(N) - \Theta_0}{N}$.

④ a) $x_{n+1} = F(x_n) = x_n + \Omega - \frac{K}{2\pi} \sin 2\pi x_n$ (circle map w/out mod 1)

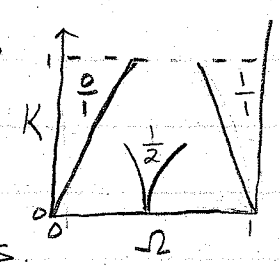
Winding number: $w = \lim_{n\to\infty} \frac{x_n - x_0}{n}$

A winding number $\frac{p}{q}$ implies that once you've converged on the values of x in the cycle, every time you take q steps (i.e., iterations) you'll go around the circle exactly p times (returning to where you started). In other words, if $w = \frac{p}{q}$, $F^q(x) = x + p$ [using converged value of x]

[p, q are integers in this discussion]

With $\underline{q=1}$, $F(x) = x + \Omega - \frac{K}{2\pi} \sin 2\pi x = x + p$

$\hookrightarrow \Omega = \frac{K}{2\pi} \sin 2\pi x + p$

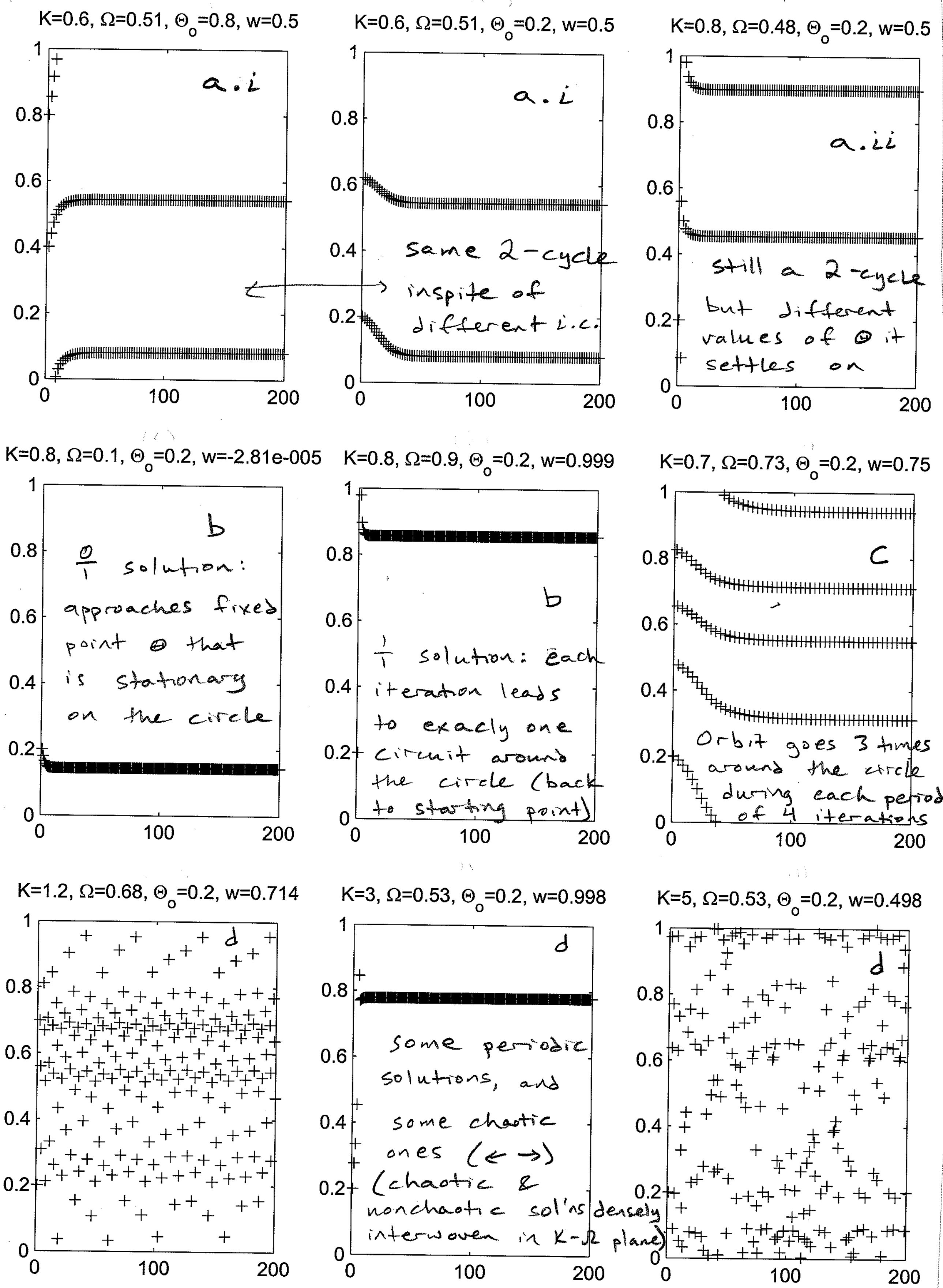


For every (Ω, K) with the $\frac{p}{1}$ Arnold tongue, there is an x satisfying this.

For $\underline{p=0}$, edge of tongue has x that gives max Ω, which is $\sin 2\pi x = 1 \Rightarrow \boxed{\Omega = \frac{K}{2\pi}}$

For $p=1$, edge of tongue has x giving min Ω, which is $\sin 2\pi x = -1 \Rightarrow \boxed{\Omega = 1 - \frac{K}{2\pi}}$

(3) cont'd


K=0.6, Ω=0.51, Θ_o=0.8, w=0.5
a.i
K=0.6, Ω=0.51, Θ_o=0.2, w=0.5
a.i
same 2-cycle inspite of different i.c.
K=0.8, Ω=0.48, Θ_o=0.2, w=0.5
a.ii
still a 2-cycle but different values of Θ it settles on
K=0.8, Ω=0.1, Θ_o=0.2, w=-2.81e-005
b
0/1 solution: approaches fixed point Θ that is stationary on the circle
K=0.8, Ω=0.9, Θ_o=0.2, w=0.999
b
1/1 solution: each iteration leads to exacly one circuit around the circle (back to starting point)
K=0.7, Ω=0.73, Θ_o=0.2, w=0.75
c
Orbit goes 3 times around the circle during each period of 4 iterations
K=1.2, Ω=0.68, Θ_o=0.2, w=0.714
d
K=3, Ω=0.53, Θ_o=0.2, w=0.998
d
some periodic solutions, and some chaotic ones (← →) (chaotic & nonchaotic sol'ns densely interwoven in K-Ω plane)
K=5, Ω=0.53, Θ_o=0.2, w=0.498
d

(4) b) Consider $\frac{p}{q} = \frac{1}{2}$. Now $F^2(x) = x+1$.

[Note: $F^2(x)$ means $F(F(x))$].

$$F^2(x) = x + 2\Omega - \frac{K}{2\pi}\sin 2\pi x - \frac{K}{2\pi}\sin\left[2\pi\left(x+\Omega-\frac{K}{2\pi}\sin 2\pi x\right)\right] = x+1$$

We're considering small K, so let

$\Omega \simeq \Omega_0 + \Omega_1 K + \Omega_2 K^2$ and expand in powers.

$$2\Omega_0 + 2\Omega_1 K + 2\Omega_2 K^2 - \frac{K}{2\pi}\sin 2\pi x - \frac{K}{2\pi} G - 1 = 0$$

Find orders of K in G by Taylor expanding about $K=0$

$$G = \sin 2\pi(x+\Omega_0) - K\cos 2\pi(x+\Omega_0)(2\pi\Omega_1 - \sin 2\pi x) + O(K^2)$$

$O(K^0)$: $2\Omega_0 - 1 = 0 \Rightarrow \Omega_0 = \frac{1}{2}$

$O(K^1)$: $\left(2\Omega_1 - \frac{1}{2\pi}\sin 2\pi x - \frac{1}{2\pi}\sin 2\pi(x+\Omega_0)\right)K = 0$

since $\Omega_0 = \frac{1}{2}$, $\sin 2\pi(x+\frac{1}{2}) = -\sin 2\pi x \Rightarrow \Omega_1 = 0$

$O(K^2)$: $\left(2\Omega_2 - \frac{1}{2\pi}\cos 2\pi(x+\Omega_0)(2\pi\Omega_1 - \sin 2\pi x)\right)K^2 = 0$

since $\Omega_0 = \frac{1}{2}$, $\cos 2\pi(x+\frac{1}{2}) = -\cos 2\pi x$, $\Omega_1 = 0$

$$2\Omega_2 - \frac{1}{2\pi}\cos 2\pi x \sin 2\pi x = 2\Omega_2 - \frac{1}{4\pi}\sin 4\pi x = 0 \Rightarrow \Omega_2 = \frac{1}{8\pi}\sin 4\pi x$$

$$\Omega = \frac{1}{2} + \frac{K^2}{8\pi}\sin 4\pi x + O(K^3)$$

The tongue boundary is at $\sin 4\pi x = \pm 1$, so it's

$$\boxed{\Omega = \frac{1}{2} \pm \frac{K^2}{8\pi} + O(K^3)}$$