Homework #10
Nonlinear dynamics and chaos

1. Prove the symplectic condition for Hamiltonian systems (see Ott).

2. (Optional, only if you have some background in classical mechanics, see Goldstein or some other such textbook)

   (a) Show that the new Hamiltonian $\bar{H}$ obtained via a generating function S is given by $\bar{H}(\bar{p},\bar{q},t) = H(p,q,t) + \partial S/\partial t$.

   (b) (Joseph McCauley, "Classical Mechanics", p. 406, q. 1) Show that the transformation

   $$Q = p + iaq; \qquad P = (p - iaq)/(2ia)$$

   is canonical and find the generating function (remember that the generating function may be one of $F_1(q,Q,t)$, $F_2(q,P,t)$, $F_3(p,Q,t)$, $F_4(p,P,t)$). Use the transformation to formulate the linear harmonic oscillator problem in terms of Q,P.

   (c) (Joseph McCauley, "Classical Mechanics", p. 406, q. 2) For which constants α and β is the transformation

   $$Q = \alpha p/x; P = \beta x^2$$

   canonical? Find the generating function.

3. *Standard map:* Plot the "transition to chaos" of the standard map: vary the nonlinearity parameter of the map to see the breakup of quasi-periodic tori into smaller tri and chaotic regions. Find also the nonlinearity for which one of the smaller tori formed breaks again into yet smaller ones. Plot all stages demonstrating this double tori breakup.
   Note that you need to use many different initial conditions for the map to observe the different tori and chaotic regions. Adjust the density of the initial conditions and the number of iterations plotted to obtain a good quality of plots.

4. **Extra credit:** (Ott 1st ed, p 263, q 3) Consider a magnetic field in a plasma given by

   $$\mathbf{B}(x,y,z) = B_0\mathbf{z}_0 + \nabla \times \mathbf{A}$$

   where B_0 is a constant, and the vector potential $\mathbf{A}$ is purely in the z-direction, $\mathbf{A} = A(x,y,z)z_0$. Denote the path followed by a field line as

   $$x(z) = x(z)\mathbf{x}_0 + y(z)\mathbf{y}_0 + z\mathbf{z}_0$$

   where $\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0$ are unit vectors. Show that the equations for $x(z)$ and $y(z)$ are in the form of Hamilton's equations, where z plays the role of time and $A(x,y,z)/B_0$ plays the role of the Hamiltonian.