APM203 Homework 10 solutions

Problem # 1

Prove the symplectic condition for Hamiltonian systems. Following OH:

$$\delta x^{+} \cdot S_N \cdot \delta x' = \delta p \cdot \delta q' - \delta q \cdot \delta p'$$

$$(*) \quad \frac{d}{dt}\, \delta x^{+} \cdot S_N \cdot \delta x' = \frac{d\,\delta x^{+}}{dt} \cdot S_N \cdot \delta x + \delta x^{+} \cdot S_N \cdot \frac{d\,\delta x'}{dt}$$

Now, using $\frac{d\vec{x}}{dt} = \vec{F}(\vec{x}, t)$

and $\delta\vec{x} = \vec{x} - \vec{\tilde{x}} \Rightarrow \frac{d\,\delta\vec{x}}{dt} = \frac{d\vec{x}}{dt} - \frac{d\vec{\tilde{x}}}{dt}$

$$= \vec{F}(\vec{x} + \delta\vec{x}) - \vec{F}(\vec{x})$$

$$= \vec{F}(\vec{x}) + \frac{d\vec{F}}{d\vec{x}}\,\delta\vec{x} - \vec{F}(\vec{x})$$

$$= \frac{d\vec{F}}{d\vec{x}}\,\delta\vec{x}$$

thus $$(*) = \left(\frac{d\vec{F}}{d\vec{x}} \cdot \delta\vec{x}\right)^{+} \cdot S_N \cdot \delta x' + \delta x^{+} \cdot S_N \cdot \frac{dF}{dx} \cdot \delta x'$$

$$= \delta\vec{x}^{+} \cdot \left(\frac{d\vec{F}}{d\vec{x}}\right)^{+} \cdot S_N \cdot \delta x' + \delta x^{+} \cdot S_N \cdot \frac{dF}{dx} \cdot \delta x'$$

From $(AB)^{+} = B^{+}A^{+}$.

$$= \delta\vec{x}^{+} \cdot \left[\left(\frac{d\vec{F}}{d\vec{x}}\right)^{+} \cdot S_N + S_N \cdot \left(\frac{d\vec{F}}{d\vec{x}}\right)\right] \cdot \delta\vec{x}'$$

$$= \delta\vec{x}^{+} \cdot \left[\left(S_N \cdot \frac{\partial^2 H}{\partial x \partial x}\right)^{+} \cdot S_N + S_N \cdot \left(S_N \cdot \frac{\partial^2 H}{\partial x \partial x}\right)\right] \cdot \delta x'$$

From

$$\left\{ \vec{F} = S_N \cdot \frac{\partial H}{\partial \vec{x}} \right\}.$$

thus, $= \delta \vec{x}^{\,t} \cdot \left[\left(\frac{\partial^2 H}{\partial^2 x} \right)^{t} \cdot S_N^{t} \cdot S_N + S_N \cdot S_N \cdot \left(\frac{\partial^2 H}{\partial x^2} \right) \right] \cdot \delta \vec{x}$

But $S_N = \begin{bmatrix} 0_N & -I_N \\ I_N & 0_N \end{bmatrix} \Rightarrow S_N^{t} = -S_N$

thus $S_N \cdot S_N = -I_{2N}$ while $S_N \cdot S_N^{t} = I_{2N}$.

Furthermore, $\frac{\partial^2 H}{\partial \vec{x} \partial \vec{x}}$ is a symmetric matrix by virtue of equal mixed derivatives e.g. $\frac{\partial^2 H}{\partial p \partial q} = \frac{\partial^2 H}{\partial q \partial p}$ etc.

Putting all of this together, we arrive at symplectic cond.

Problem #2 (Optional)

(a) If New variables $\bar{q}$ and $\bar{p}$ are to be canonical they must satisfy Hamilton's principle:

$$\delta \int_{t_1}^{t_2} (\bar{p} \cdot \dot{\bar{q}} - K(\bar{p}, \bar{q}, t))\, dt = 0$$

N-vectors

New Hamiltonian.

(or as Goldstein observes... Kamiltonian?)

while the old coordinates satisfy their Hamilton's principle

$$\delta \int_{t_1}^{t_2} (p \cdot \dot{q} - \mathcal{H}(p, q, t))\, dt = 0$$

Apart from an unimportant scaling factor these two are simultaneously valid if

$$p\dot{q} - \mathcal{H} = \bar{p}\cdot\dot{\bar{q}} - \mathcal{K} + \frac{dF}{dt}$$

since this contributes nothing at fixed endpoints upon variation δ of path.

thus $$p\cdot\dot{q} - \mathcal{H} = \bar{p}\cdot\dot{\bar{q}} - \mathcal{K} + \frac{\partial F}{\partial t} + \frac{\partial F}{\partial q}\dot{q} + \frac{\partial F}{\partial \bar{q}}\dot{\bar{q}}$$

where we've picked a generating function of 1^{st} type. This is inconsequential for our purposes since we focus on time variable, always present: matching $\dot{q}$'s and $\dot{\bar{q}}$'s by using $\frac{d\vec{x}}{dt} = S_N \cdot \frac{\partial H}{\partial \vec{x}}$, we get

$$\boxed{\mathcal{K} = \mathcal{H} + \frac{\partial F}{\partial t} \quad \text{as desired}}$$

part b.

The generating function will not depend explicitly on time since Q, P are time-independent. Thus, $\mathcal{H} = K$. Furthermore, let's look at $F_1(q, Q)$:

$$p\dot{q} = P\dot{Q} + \frac{\partial F_1}{\partial q}\dot{q} + \frac{\partial F_1}{\partial Q}\dot{Q}$$

now $P = -\frac{\partial F_1}{\partial Q}$

$p = \frac{\partial F_1}{\partial q}$

now, $$Q = p + iaq \qquad \text{(i)}$$

$$P = (p - iaq)/(2ia) \qquad \text{(ii)}$$

$$P = \frac{p - iaq}{2ia} = \frac{(Q - iaq) - iaq}{2ia} = \frac{Q - 2iaq}{2ia} = -\frac{\partial F_1}{\partial Q}$$

$$\Rightarrow \frac{\partial F_1}{\partial Q} = q - \frac{Q}{2ia} \quad \text{or} \quad F_1 = qQ - \frac{Q^2}{4ia} + f_1(q)$$

and $\quad p = Q - iaq \Rightarrow \quad \dfrac{\partial F_1}{\partial q} = Q - iaq$

or $\quad F_1 = qQ - \dfrac{iaq^2}{2} + f_2(Q)$

$$\Rightarrow F_1(q, Q) = qQ - \frac{Q^2}{4ia} - \frac{iaq^2}{2}$$

Existence of F_1 $\Rightarrow$ transformation is canonical ✓

Rewrite $\mathcal{H}(p,q) = K(P,Q)$.

$$\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2$$

now, $\quad p = \frac{1}{2}(Q + 2iaP)$

$\quad\quad\quad q = \frac{1}{2ia}(Q - 2iaP)$

thus $\quad \mathcal{H} = K(Q,P) = \frac{1}{2m}\left(\frac{1}{4}\right)(Q + 2iaP)^2 - \frac{1}{2}m\omega^2\left(\frac{1}{4a^2}\right)(Q - 2iaP)^2$

to simplify, which is really the purpose of these transformations,

let $a = m\omega$. This was arbitrary from the start.

$$\rightarrow K(Q,P) = \frac{1}{8m}\left(\cancel{Q^2} + 2\cdot 2iaQP - \cancel{4a^2P^2} - \cancel{Q^2} + 2\cdot 2iaQP + \cancel{4a^2P^2}\right)$$

$$K(Q,P) = \frac{1}{8m}\left(8imw\,QP\right) = iw\,QP$$

Solve the linear harmonic oscillator problem in terms of Q & P:

$$-\frac{\partial K}{\partial Q} = \dot{P} \quad ; \quad \frac{\partial K}{\partial P} = \dot{Q}$$

i.e. $$\dot{P} = -iwP \quad ; \quad \dot{Q} = iwQ$$

or $$P(t) = e^{-iwt} P_0 \quad ; \quad Q(t) = e^{iwt} Q_0 \, .$$

note $$Q(t)\,P(t) = Q_0 P_0$$ as expected from conservation of energy.

part c. $$Q = \frac{\alpha p}{x} \quad ; \quad P = \beta x^2$$

look for generating function $F_1(x, Q)$

$$P = -\frac{\partial F_1}{\partial Q} \implies -\frac{\partial F_1}{\partial Q} = \beta x^2$$

thus $$F_1 = -\beta x^2 Q + f_1(x)$$

and $$p = \frac{\partial F_1}{\partial x} \implies \frac{\partial F_1}{\partial x} = \frac{1}{\alpha} x Q$$

$$F_1 = \frac{1}{\alpha} \cdot \frac{x^2}{2} Q + f_2(Q)$$

thus, if $\frac{1}{2\alpha} = -\beta$, transformation is canonical.

And the generating function is $\boxed{F_1(q,Q) = \frac{1}{2} x^2 Q}$

if we choose $\alpha = +1$, $\beta = -\frac{1}{2}$, and $f_1 = f_2 = 0$.

$\boxed{\text{Problem \#3}}$

Numerics were done well by everyone, so I'll skip this problem.

$\boxed{\text{Problem \#4}}$ (Extra Credit)

Magnetic field plasma given by

$$\vec{B}(x,y,z) = B_0 \vec{z}_0 + \nabla \times \vec{A}$$

$$\vec{A} = A(x,y,z)\,\vec{z}_0 \Rightarrow \nabla \times \vec{A} = (\partial_y A, -\partial_x A, 0)$$

and $\vec{B} = (\partial_y A, -\partial_x A, B_0)$

Denoting path followed by a field line $\vec{B} = \text{const.}$

as $x(z) = x(z)\vec{x}_0 + y(z)\vec{y}_0 + z\vec{z}_0$

Field line satisfies

normalized to agree with z-component.

$$\left(\frac{dx}{dz}, \frac{dy}{dz}, 1\right) = \frac{1}{B_0}(\partial_y A, -\partial_x A, B_0)$$

$$\Rightarrow \left\{ \frac{dx}{dz} = \partial_y\left(\frac{A}{B_0}\right) \qquad \frac{\partial y}{\partial z} = -\partial_x\left(\frac{A}{B_0}\right) \right\}$$

"time" "Hamiltonian"