Homework #7
Nonlinear dynamics and chaos

1. **Quasi-periodicity route to chaos:** Solve this problem either by running the Matlab program `pendulum.m` on the course home page for simulating a periodically forced and damped pendulum, *or* just by thinking about this problem and coming up with the right answer...

   (a) Choose some value for all model parameters, and increase the amplitude of the forcing from zero in order to see how the pendulum behavior changes from damped to chaotic. Explain what regimes are encountered on the way.

   (b) Plot the time series of $\theta(t)$ (either from actually running the Matlab program, or just schematically) from the solution of the forced and damped pendulum equation, for each of the regimes encountered from the damped to the chaotic regimes.

   (c) Plot a Poincare section based on sub-sampling the time series for $\theta(t)$ every period of the forcing. Do this for the different regimes that are encountered as the forcing amplitude increases from damped to chaotic regimes. Explain why the Poincare section looks the way it does for each of these regimes.

   (d) What would a quasi-periodic regime mean in this problem? Do you see it in your experiment as the forcing amplitude is increased? Why?

2. **Length of non-chaotic intervals for type III intermittency:** Find a map that appropriately describes a type III trapping region in the intermittency route to chaos. Explain why this map is the right one (can use Schuster or Ott...). Use $x_{n+2} - x_n \approx dx/dn$ (why is it appropriate to use this approximation in the trapping region? why $n+2$ rather than $n+1$?) and show that the length of non-chaotic intervals in this case behaves like ε^{-1}. Plot the iterates of the map in the trapping region.

3. **Analytic calculation of Arnold's tongues in the circle map near $K = 0$:** Do *only one* of the following two problems:

   (a) Consider the circle map

   $$x_{n+1} = F(x_n) = x_n + \Omega - \frac{K}{2\pi}\sin(2\pi x_n) \qquad \text{(mod 1)}.$$

   It can be shown that the rotation (winding) number is p/q if and only if

   $$F^q(x) - (x+p) = 0.$$

   First, test this numerically for some two different values of (p,q). Next, use this relation to show that the edges of the Arnold tongues $q = 1$ and $p = 0$ or $p = 1$ are at $\Omega = K/(2\pi)$ (for $p = 0$) and $\Omega = 1 - K/(2\pi)$ (for $p = 1$).

   (b) **Challenge/ Extra credit:** Using the same approach as the previous question, show that the boundary of the Arnold tongues $p = 1$ and $q = 2$ for small K is given by $\Omega = \frac{1}{2} \pm \frac{K^2}{8\pi}$.