APM 203 Homework #7

Solutions

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Problem #1

$$\ddot{\theta} + b\dot{\theta} + \omega_0^2 \sin\theta = \gamma_1 \cos \omega t$$

The following traces (taken from d'Humières et. al 1982) illustrate experimental (using Josephson junction) experimental values of $\langle\dot{\theta}\rangle$ vs. ω/ω_0 for various values of forcing parameter γ_1. (or equivalently $(\langle F(t)^2\rangle)^{1/2}$.)

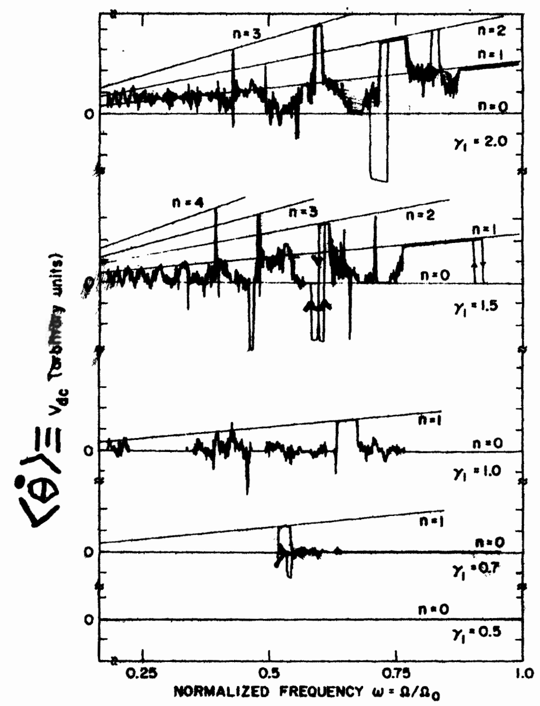


FIG. 1. Voltage fluctuations of the simulator circuit after dc as a function of the driving frequency for various driving amplitudes. Note pattern of phase-locked and -unlocked states. Straight lines indicate phase-locked steps satisfying the relationship $V = nk^{-1}\omega$.

← still too small, but chaos allows for fluctuations.

← here, γ_1 is smaller than critical torque to swing once around

Note that hysteresis loops are visible (marked by arrows) on these traces. How do these come about? Consider first γ_1 rather small, say $\gamma_1 \sim 0.7$ (second to last trace in figure) then eqn. of motion is about:

$$\ddot{\theta} + b\dot{\theta} + \omega_0^2\theta - \frac{\epsilon}{6}\omega_0^2\theta^3 = \gamma_1 \cos \omega t. \rightarrow \text{DUFFING OSCILLATOR,}$$

when ϵ is put in to label small term.

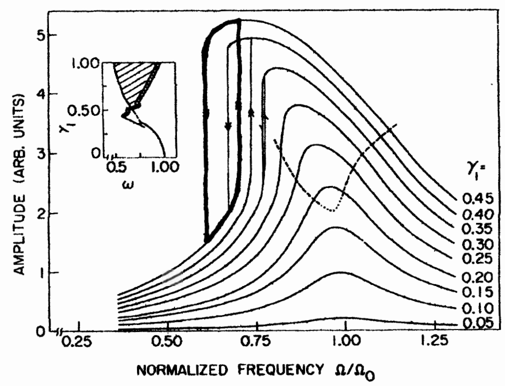
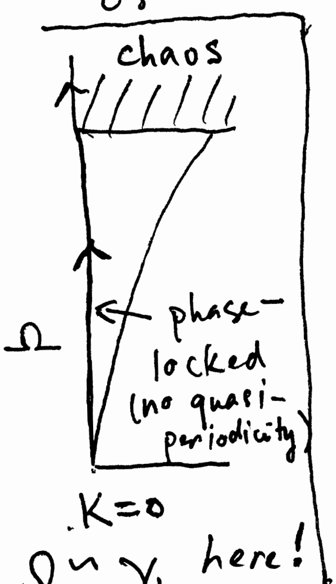


FIG. 4. Observed filtered (fundamental) amplitude response of the simulator circuit for various driving amplitudes. Dashed line shows domain of broken symmetry in the pendulum motion. Insert shows the observed bifurcation diagram similar to that reported in Ref. 1.

Here, I've marked a hysteresis loop in purple. These amplitude resonance diagrams may be derived analytically by truncating the perturbation series and keeping only a few subharmonics (e.g. see José & Saletan p. 381–390)

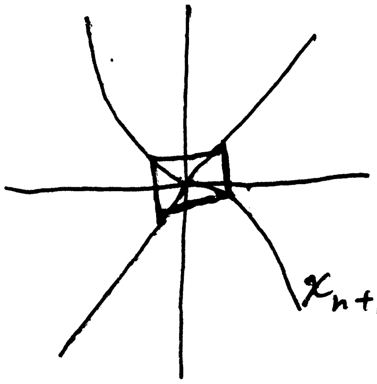
One can understand the behavior of this driven oscillator as follows: If the pendulum cannot swing once around with forcing amplitude imparted on it, we have a phase-locked state (albeit distorted – the pendulum breaks its spatial symmetry) and no chaos (Duffing oscillator). A period-doubling cascade is initiated as $\gamma_1 \sim 0.7$ as the system leaves the phase-locked tongue (left diagram) for other phase-locked tongues in the cross-hatched region. This amounts to the pendulum flipping its rotation rate $\langle \dot{\theta} \rangle$ at more and more subharmonics. Quasi-periodic phases are impossible for $b \neq 0$, due to the absence of a constant torque, which would provide a second forcing parameter. In the latter case we would expect $\langle \dot{\theta} \rangle$ to be of incommensurate frequency with ω for $b \neq 0$, especially at low γ_1 (i.e. outside of Arnold tongues).



Problem # 2

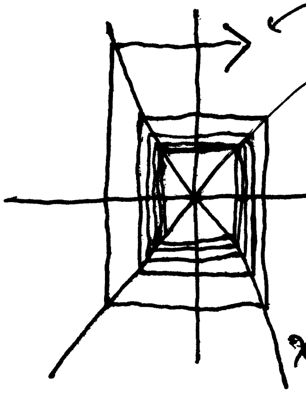
Intermittency Mechanism III. Type III intermittency, we've learned in class, is associated with a sub-critical period-doubling bifurcation and occurs in Rayleigh-Bénard convection close to the Rayleigh number $R/R_c = 416.5$ (Dubois et. al. 1983).

Below the threshold intermittency value R_i, we have a stable limit cycle. Slightly above R_i, periodic bursts of chaotic behavior is observed, followed by long laminar (periodic) phases where the system is trapped in the ghost of the fixed point.



$\epsilon < 0$

$$x_{n+1} = -(1+\epsilon)x_n + \alpha_1 x_n^2 + \alpha_2 x_n^3.$$



$\epsilon \gtrsim 0$

here the mapping leaves the intermittency region only to be reinserted by chaos (one assumes there are higher-order terms of course, and that the map is non-invertible!)

$$x_{n+1} = -(1+\epsilon)x_n + \alpha_1 x_n^2 + \alpha_2 x_n^3$$

To analyze the map, we investigate its second iterate (this only scales time by a constant factor - and will not affect our results!)

$$x_{n+2} = -(1+\epsilon)\left[-(1+\epsilon)x_n + \alpha_1 x_n^2 + \alpha_2 x_n^3\right] +$$
$$+\alpha_1\left[(1+\epsilon)^2 x_n^2 - 2\alpha_1(1+\epsilon)x_n^3\right] - \alpha_2(1+\epsilon)x_n^3 + \ldots$$

drop terms smaller than $O(\epsilon)$. The $O(\epsilon)\, x_n^2$ term may also be dropped since rescaling can eliminate it (complete the square, shift x_n, etc.) and it won't affect our calculations...

$$x_{n+2} = (1+2\epsilon)x_n + \alpha x_n^3 + \ldots \qquad \alpha = -2(\alpha_2 + \alpha_1^2).$$

$$\frac{dx}{dn} \approx 2\epsilon x + \alpha x^3 \implies \text{pitchfork bifurcation!}$$

note, we require $\alpha > 0$ for subcritical behavior $\Leftarrow$ chaos.

Scaling x by $\sqrt{2\epsilon/\alpha u}$ and n by $k(2\epsilon)^{-1}$,

$$\frac{du}{dk} = u(1+u^2) \qquad x = u\sqrt{\frac{2\epsilon}{\alpha}}, \quad n \approx \frac{k}{2\epsilon}$$

This is of universal form (same as Type III intermittency)

thus, $n = \frac{k}{2\epsilon} \sim \frac{1}{\epsilon}$ indicates that the laminar phase period scales as $1/\epsilon$.

Problem # 3

$$W = \lim_{n\to\infty} \frac{F(x_n) - x_0}{n} \to \frac{p}{q}$$

iff $\quad F^q(x) - (x+p) = 0 \quad$ (see Nayfeh et al.) for proof.

→ sufficiency: if the above is satisfied, $F^q(x_0) = x_0 + p$ and $F^{mq}(x_0) = x_0 + mp$. Let $n = mq$. then

$$W = \lim_{n\to\infty} \frac{F^n(x_0) - x_0}{n} = \lim_{m\to\infty} \frac{F^{mq}(x_0) - x_0}{mq} = \frac{p}{q}$$

(a) Find region of mode-locking (Arnold tongue) for $q=1$ and $p=0$ or $p=1$. (No full cycle, one full cycle). then, for $p=0$,

$$F(x) - x = 0 \Rightarrow x + \Omega - \frac{K}{2\pi}\sin 2\pi x = x$$

or $\quad \Omega = \frac{K}{2\pi} \sin 2\pi x \qquad$ (real x, K, Ω; $\Omega, K \geq 0$.)

but $|\sin(z)| \leq 1 \Rightarrow \Omega \leq \frac{K}{2\pi}$ i.e.

$\Omega = \frac{K}{2\pi}$ defines boundary of this mode. (Surprisingly fat!)

similarly, for $p=1 \quad \Omega = 1 - \frac{K}{2\pi}$

(b) Now consider the mode-locked tongue $p=1, q=2$.

then:

$$F^2(x) = x+1, \quad \text{for some } x \in [0,1].$$

thus, $F^2(x) = x + 2\Omega - \epsilon \sin 2\pi x - \epsilon \sin\left[2\pi\left(x + \Omega + \epsilon \sin 2\pi x\right)\right]$

where $\epsilon = \dfrac{K}{2\pi}$ and we consider $\epsilon << 1$.

Let $\Omega = \frac{1}{2} + \beta$. $\frac{p}{q} = \frac{1}{2}$ iff

$$2\beta - \epsilon \sin 2\pi x - \epsilon \sin\left[2\pi\left(x + \tfrac{1}{2} + \beta - \epsilon \sin 2\pi x\right)\right] = 0$$

or $$2\beta - \epsilon \sin 2\pi x + \epsilon \sin\left[2\pi\left(x + \beta - \epsilon \sin 2\pi x\right)\right] = 0$$

Expand last term in this eqn. for small ϵ and β:

$$2\beta - \epsilon \sin 2\pi x + \epsilon \sin 2\pi x + 2\pi\epsilon\left(\beta - \epsilon \sin 2\pi x\right)\cos 2\pi x + O(\epsilon^3, \epsilon\beta^2, \epsilon^2\beta) = 0$$

or $$\beta - \tfrac{1}{2}\epsilon^2 \pi \sin 4\pi x + \pi\epsilon\beta \cos 2\pi x + \ldots = 0$$

Hence $\beta = \frac{1}{2}\epsilon^2\pi \sin 4\pi x + O(\epsilon^2)$

thus $|\beta| \leq \frac{1}{2}\epsilon^2\pi$ or $\boxed{\Omega = \frac{1}{2} \pm \frac{K^2}{8\pi}}$

at $K=1$, this occupies a width $\frac{1}{4\pi}$ (half of $q=1$ tongues.) on the devil's staircase.