APM 203 Homework 5, Solutions

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Everyone did excellently on problems 2 and 3. So I'll provide solutions for 1 and the challenge problem, 4.

Problem #1:

$$\dot{X} = -\sigma(X - Y)$$
$$\dot{Y} = -XZ + rX - Y$$
$$\dot{Z} = XY - bZ$$

Fixed points of Lorenz Equations:

$$(0,0,0) \quad \text{and} \quad (\pm\xi, \pm\xi, r-1) \qquad \xi \equiv \sqrt{(r-1)b}$$

the origin is stable for $0 \leq r < 1$ and bifurcates into two stable spirals $C_\pm$ above at $r = 1$. The spirals become unstable at $r = r_H$ when ~~their~~ eigenvalues cross the imaginary axis. i.e. $\lambda_\pm = \pm i\lambda_0$ (λ_0 real). where $\lambda_\pm$ is evaluated at $r = r_H$.

Linearization about $C_\pm$:

$$J_\pm = \begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & \mp\xi \\ \pm\xi & \pm\xi & -b \end{pmatrix}$$

characteristic eqn. (independent of $\pm$ $\Rightarrow$ both $C_\pm$ become unstable spirals at same r_H.)

$$\lambda^3 + (1+b+\sigma)\lambda^2 + (b + b\bar{r} + b\sigma)\lambda + 2\bar{r}b\sigma = 0$$

$\bar{r} \equiv r-1.$

at Hopf bifurcation, two conjugate pairs are pure imaginary; characteristic equation takes the form:

$$(\lambda - a)(\lambda + i\lambda_0)(\lambda - i\lambda_0) = 0$$

$$\lambda^3 - a\lambda^2 + \lambda_0^2\lambda^2 - a\lambda_0^2 = 0$$

thus, if we want a Hopf bifurcation, we must have λ^2-coefficient $\times$ λ^1-coefficient $=$ λ^0 coefficient. i.e.

$$(1+b+\sigma)(\cancel{b} + \cancel{b}\bar{r}_H + \cancel{b}\sigma) = 2\bar{r}_H\cancel{b}\sigma$$

or $$\boxed{r_H = \sigma\,\frac{3+b+\sigma}{\sigma-b-1}}$$

Problem #4

Investigate the dynamical equations:

$$\ddot{u} + \omega^2 u = (\epsilon - \alpha z)\dot{u} \qquad \tau > 0, \quad \epsilon \text{ small.}$$
$$\dot{z} + \tau z = u^2 \qquad \alpha > 0$$

for a limit cycle in the form $u = a\cos(\omega t + \beta)$.

First, let's rewrite the system as 3 1^{st}-order diff. eqns.

$$\dot{y} = (\epsilon - \alpha z) y - \omega^2 u$$
$$\dot{u} = y$$
$$\dot{z} = u^2 - \tau z$$

clearly, if ϵ were zero, the system would decay to the origin in the vicinity of the origin. As ϵ increases a tiny bit, we're interested in a possible limit cycle of small amplitude emerging from $(0,0,0)$. Let $a = O(\eta)$ where η is a book-keeping parameter, and denotes the size of lowest-order term of u. Then $u = O(\eta)$, $y = O(\eta)$ and $z = O(\eta^2)$. This means ϵ becomes important (as it increases from zero) at order $O(\eta^2)$. Thus we write $\epsilon = \eta^2 \epsilon_2$ for purpose of keeping track of terms.

Introduce slow times and perturbation series in η:

$$\frac{d}{dt} = D_0 + \eta D_1 + \eta^2 D_2 + \dots \qquad D_n \equiv \frac{\partial}{\partial T_n}.$$

$$u = \sum_{n=1}^{3} \eta^n u_n (T_0, T_1, T_2) + \dots$$

$$y = \sum_{n=1}^{3} \eta^n y_n (T_0, T_1, T_2) + \dots$$

$$z = \sum_{n=2}^{3} \eta^n z_n (T_0, T_1, T_2) + \dots$$

$$\epsilon = \eta^2 \epsilon_2$$

Insert into eqns. and separate each group of η^n terms up for $n = 3$:

$O(\eta)$:

$$D_0 y_1 = -\omega^2 u_1$$
$$D_0 u_1 = y_1$$
$$D_0 z_1 = -\tau z_1$$

$O(\eta^2)$:

$$D_1 y_1 + D_0 y_2 = \overset{0}{\cancel{\epsilon_1}} y_1 - \omega^2 u_2$$
$$D_1 u_1 + D_0 u_2 = y_2$$
$$D_1 z_1 + D_0 z_2 = u_1^2 - \tau z_2$$

$O(\eta^3)$:

$$D_2 y_1 + D_0 y_3 = \epsilon_2 y_1 - \alpha y_1 z_2 - \omega^2 u_3$$
$$D_2 u_1 + D_0 u_3 = y_3$$
$$D_1 z_2 + D_0 z_3 = 2 u_1 u_2 - \tau z_3$$

solving $O(\eta)$ equs.,

$$u_1 = A(T_1, T_2)\, e^{i\alpha T_0} + c.c. \qquad \underline{\underline{\alpha = \omega}}.$$

$$y_1 = i\omega A(T_1, T_2)\, e^{i\alpha T_0} + c.c.$$

$$z_1 = B(T_1, T_2)\, e^{-\tau T_0}$$

ultimately, we're only interested in the <u>non-decaying</u> solution.

Thus, $z_1 = 0$ (as expected from order-of-magnitude arguments above.)

Next, plug into $O(\eta^2)$ equs and set secular terms to zero:

$$\Rightarrow (D_1 A) = 0 \quad \Rightarrow \quad A = A(T_2)$$

$$u_2 = y_2 = 0$$

and
$$z_2 = \frac{2|A|^2}{\tau} + \frac{A^2}{2i\omega + \tau} \cdot e^{2i\omega T_0} + c.c.$$

Finally, plugging these results in turn into first two $O(\eta^3)$ equs, and setting secular terms to zero:

$$A' = \frac{1}{2}\epsilon_2 A - \frac{\alpha}{\tau} A|A|^2 + \frac{1}{2}\alpha A \frac{|A|^2}{2i\omega + \tau} \qquad A' \equiv \frac{dA}{dT_2}.$$

with $A = \frac{1}{2} a e^{i\beta}$ (with c.c. this is just $a\cos(\beta)$. and $u_1 = a\cos(\omega t + \beta)$ as required.)

$$\dot{a} = \frac{1}{2}\cdot\epsilon\cdot a - \frac{\alpha}{8\tau}\left(\frac{8\omega^2 + \tau^2}{4\omega^2 + \tau^2}\right) a^3$$

$$\dot{\beta} = -\frac{\alpha}{4}\,\frac{\omega}{\tau^2 + 4\omega^2}\, a^2$$

Thus, a limit cycle of radius $\left(\frac{4\tau}{\alpha}\cdot\frac{4\omega^2+\tau^2}{8\omega^2+\tau^2}\right)^{1/2}\epsilon^{1/2}$ appears with $\epsilon > 0$ that travels clockwise through phase plane, with angular velocity $O(\epsilon)$.