Mathematical modeling

Applied Mathematics 115

(Spring 2007)

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Workshop (section) times and place: first meetings will be on the 6th and 7th of February, 4:00–6:00 pm; CAD-1 classroom in the Maxwell-Dworkin basement.

Day & time: Tue-Thu 11:30-1pm.

Location: Maxwell Dvorkin 125;

1st meeting: Thursday Feb 1st.

Bibliography: material from several textbooks and other sources will be used; the source materials for all lectures, including Matlab programs used in class, may be found here. Follow links below for the specific source material for each lecture.

Web: http://isites.harvard.edu/icb/icb.do?keyword=k9215

This document: http://www.deas.harvard.edu/climate/eli/Courses/2007spring_a/detailed-syllabus-apm115_modelling.pdf

Announcements Last updated: April 19, 2007.

Feel free to write call or visit us with any questions.

Workshops: 1, 2, 3, 4, 5,

Projects: first project description,
second project description,
third project description,
feedback on projects and presentations, please look at it carefully before and while working on your project and presentation

1 Outline

Mathematical models are ubiquitous, providing a quantitative framework for understanding, prediction and decision making in nearly every aspect of life, ranging from timing traffic lights, to controlling the spread of disease, to weather, climate or earth quakes, to economic forecasting. This course provides an introduction to modeling through in depth discussion of a series of real examples.

2 Administrative

Prerequisites: Applied Mathematics 21a and 21b, or Mathematics 21a and 21b or permission of instructor. Knowledge of some programing language is helpful but not necessary as we'll be introducing Matlab for those with no previous experience.

Workshop: In addition to the lectures, the class will have a weekly computer workshop, in which students will have the opportunity to implement and explore the models described in class on a computer. The workshops will be entirely self contained, the goal being that every student will have successfully implemented the model by the end of the workshop. Regular times for workshops will be scheduled at the beginning of the semester.

Computer Skills: No computer skills are assumed for this class. Computer labs will be run in MATLAB, and as part of the course, students will therefore gain facility with this package. Students should download and install the MATLAB system on their computers from the FAS software site (see the WWW links button on the main course page). We will schedule a crash introduction to MATLAB early next week. Additional instruction will be given in the computer workshops.

Homework: Homework assignments, in the form of model development, will be assigned roughly every two weeks. These assignments will be completed by 3 person teams (to be formed in the first workshop). These models will further explore the models discussed in class. Each team will focus on a different extension of the models, and will present their findings in a subsequent class.

Final Project: During roughly the last third of the semester, each individual will carry out a final project investigating a new model or carrying out a significant extension of a model discussed in class. Student creativity in selecting and refining the project topic will be important in evaluation of the project. During this time period, homework will not be assigned. Class time will be spent discussing "progress reports" on the projects.

Grading: Attendance in lectures and workshops is mandatory and will be part of the final grade; class Participation will also be an important consideration. Workshops 20%; Homework (student group projects) 30%; Final Project 50%;

3 An evolving syllabus

This list of topics to be covered will evolve and become more specific and detailed during the course. Follow links to see the source material and Matlab demo programs used for each lecture. We'll be building a toolbox of modeling concepts and methods during the course, and the list below indicates the toolbox items provided by each of the topics.

L1: Introduction, overview this and this.

L2-3: POPULATION DYNAMICS, SINGLE SPECIES: [**toolbox concepts:** ODEs; fixed points, linear stability of fixed points, numerical solution of ODEs, Markov process, stochastic equations, stochastic simulation, equations for probability distribution function].

- *A single species with limited resources, deterministic approach:* logistic equation, geometric approach, linearized stability analysis (Strogatz, 1994, 2.3, 2.4 p 21-25, here).
- *A single species, stochastic approach:* probability, Markov process, simulating stochastic process and calculating the pdf numerically, (Renshaw, 1991, 3.2.3, pp 55-57; 3.4, p 59-61, here); master equation, solving directly for the pdf (3.1, p 46-48);
- stochastic_logistic.m.

L4-5: POPULATION DYNAMICS, COMPETITION OF SPECIES: [**toolbox concepts:** phase plane, oscillations vs limit cycles, model validation; maps as dynamical systems, their fixed points, oscillations and chaos]

- *Two competing species:* deadly survival struggle between sheep and Rabbits (Strogatz, 1994, 6.4, p 155-159), or parts of pp 1, 2, 5 in notes. Sheep and rabbits equations, intro to phase plane, fixed points, stability, limit cycles. Linearization in 2d, quiver_sheep_rabbits.m.
- *Predator-prey oscillations:* Motivation: Lynx and snowshoe hare in Canada. Lotka-Volterra equations (eqn 4.72, Mesterton-Gibbons, 1988, , section 4.12, p 154-157, here), (or from wikipedia) and an improvement resulting in a limit cycle (10.2, p 389-393), predator_prey.m. An interesting twist: model validation using Lynx and snowshoe hare observations.
- *Logistic map:* fixed points, stability, oscillations, chaos. Why is the logistic map behavior so much richer than of the continuous logistic equation, Poincare section and the relation between maps and continuous systems. (Strogatz, 1994, 10.0-10.4, p 348-363, here).
- *Examples of projects:* bugs, birds and leaves from Strogatz; love affairs;

- [Assignments of first group projects]

L6-7: CLIMATE: [**toolbox concepts:** multiple equilibria, bifurcations, hysteresis, catastrophes, stochastic differential equations, white noise, red noise, Fourier transform, spectrum, AR(n) Markov process, variance, autocorrelation]

- *Energy balance models, from greenhouse warming to snowball earth:* Background: black body radiation, solar radiation, long wave radiation, albedo, greenhouse effect, emissivity, schematic energy budget for the atmosphere. A zero-dimensional energy balance model including the ice albedo feedback; results: bifurcations, catastrophes (section 9.7 from Aarnout van Delden's slides 1-9. Implications: snowball earth, letter to the president; energy_balance_0d.m.
- *A stochastic model of Climate variability:* Motivation: weather vs climate, fast vs slow variability; spectrum, white noise, red noise, Hasselmann's stochastic model, simulating and then solving exactly in Fourier space, solving for the climate temperature spectrum, (notes) Hasselmann.m; an equivalent discrete Markov process (AR(1)), variance and autocorrelation (notes). ar1.m.

- *Thermohaline circulation and the day after tomorrow:* Motivation: glacial cycles, Dansgaard Oeschger events, thermohaline circulation, day after tomorrow, trailer. Stommel's model, multiple equilibria (bi-stability) of ocean circulation, and abrupt climate change (Aarnout van Delden's slides 1-3, 8-11). Stommel.m.
- *Examples of projects:* room air conditioning; 1d energy balance model;

L8: MODELING "PHILOSOPHY": Why model? What's a good model? (as simple as possible, equations well motivated by data, aids our understanding, and able to surprise us and provide new insight/ prediction not obvious beforehand). Model validation, simple vs complex models, simulation vs modeling, optimization, stochastic vs deterministic, etc. notes.

L9-10 RENEWABLE AND EXHAUSTIBLE RESOURCES [**toolbox concepts:** the maximum principle in optimal control, singular solutions and bang-bang control, shadow prices.]

- *optimal control and the maximum principle:* (notes, and this from (p 344-352, Intrilligator, 1971))
- *renewable resources (such as fisheries):* maximum sustainable yield versus socially optimal yield; open-access fisheries and overfishing; monopoly fisheries. this from (p 28-33 Clark, 1976), Clark 47-50)
- *exhaustible resources (e.g. oil)* competitive equilibrium, monopoly equilibrium, social optimum, backstop technologies. Clark 135-142.
- *Projects:* non-linear extraction technologies, sales taxes, oil depletion allowance, exploration, oligopoly.

L11: First group project presentations

- [Assignments of second group projects]

L12-13: TRAFFIC FLOW: [**toolbox concepts:** discrete modeling, delayed ODEs, PDEs, waves, characteristics, shocks]

- *Motivation:* Think your commute is bad? it could be worse: traffic in Karachi from youtube; simple gif traffic animations; a very nice Java applet demonstrating many different traffic wave related phenomena at www.traffic-simulation.de; Local copies two 3d animations from the above are merging traffic from bird view and, driver to helicopter perspective; effects of adaptive cruise control on traffic jams;
- *Single-car approach:* deriving relation between inter-car distances and car velocity (Mesterton-Gibbons, 1988, 1.13, p 34-35, here), velocity-density relationship (2.3, p 57-58), propagation of a perturbation, driver reaction time and stability of traffic flow (2.8, p 76-83). traffic.m.
- *Macro approach:* car conservation and a kinematic wave equation (Haberman, 2003, 12.6.2, p 548-549, here), method of characteristics and graphical solution (12.6.3, p 549-551), development of discontinuities, shock waves and traffic jams (12.6.4, p 551-553), jump conditions across shock, shock velocity (p 554), finite time development of shocks and shock dynamics (p 556-557). Traffic light problem (Whitham, 1974, 3.1, p 71-72, here).

L14-15 INTERACTING AGENTS [**toolbox concepts:** Markov chains, ergodic distribution, recurrent classes, large deviations, replicator dynamics, Nash equilibrium, birth-death processes, mean field, Perron-Frobenius theorem]

notes

- *evolutionary biology* Markov chains: (this from (p 48-52 Karlin and Taylor, 1975)), Wright-Fisher model of gene frequency (Karlin-Taylor 55-56); Moran model of drift with constant selection (this from (p 98-101 Nowak, 2006)), Moran process in two-player games (Nowak 109-111). In the absence of mutations, all of these processes end up at a homogeneous state where all agents/genes are the same. Replicator dynamic as the mean field of the Moran process. Order-of-limits issue: when is the mean field a good approximation?
- *myopic best responders:* KMR model of uniform matching (this from (p 138-146 Fudenberg and Levine, 1998)) compared to Ellison's model of agents on a circle (Fudenberg-Levine 149-153). Bounds on the speed of convergence using second-largest eigenvalue and the Perron-Frobenius theorem (Karlin-Taylor 542-545): local interaction models converge much more quickly than uniform matching models do.
- *application to the repeated prisoners dilemma:* two-strategy systems, three-strategy systems, Axelrod's experiments (Axelrod, 1984).
- *other models of interacting agents:* Ising spin systems, voter models, contagion models.
- *Projects:* Simulate a stochastic adjustment process, compare the results to theoretical predictions (if available) or intuitions (if not). Run an Axelrod-style tournament between competing computer programs Simulate spatial games, as in Nowak Chapter 9.

L16: Second group project presentations

L17: Diffusion: Kurt's notes; Derivation of the 1d and 2d diffusion equations (e.g. for heat diffusion, Adam (2003), here, from the last paragraph of page 311 to end of first paragraph on page 313). As for boundary conditions, describe two alternatives: specified temperature (e.g. $T(x=0,L;t)=T_0$) and specified flux boundary conditions (e.g. $-\kappa\frac{dT(x=0,L;t)}{dx}=0$). Values of diffusion coefficients for various materials, and scaling/ dimensional arguments for diffusion time (beginning of page 311, and rest of page 313). Numerical solution by straight forward center finite differencing in space and the Euler time stepping scheme. Examples including the "melting" of an initial structure, a point source, and a rotating point source (all solved by diffusion_demo_apm115.m).

L19: Assignment of third group project (after first lecture on cellular automata).

L18&20: SPATIALLY EXTENDED DISCRETE SYSTEMS: [**toolbox concepts:** discrete modeling, cellular automata (CA), fractals, fractal dimension, percolation, phase transitions; time permitting: self organized criticality and $1/f$ behavior]
Strategy here: first teach several examples and then proceed to some fundamentals and a slightly more rigorous systematics.

- (first lecture) Motivation: first a brief forest fire demo applet from applet and then forest_fire_percolation.m and explain in detail the algorithm of this Matlab code.
- Making the forest more dynamic: adding probabilistic tree growth and lightening forest.m (Chopard and Droz, 2005, p 31-33);
- Conway's "game of life": show rules from wikipedia, show a simulation using life.m; show and work out in detail some interesting cases from wikipedia and animations from the "wikipedia commons". Discuss fixed points, periodic behavior, gliders, guns and more;
- Diffusion limited aggregation: first with learning_stage=1 (low resolution, slow) and then 0 (high resolution, fast).
- Particle dynamics and HPP rule: show Figs 2.9, 2.10 and 2.11 from Chopard-Droz pages 39-40 and run gas2.m. The HPP rule simulation of gas particles in a two-chamber container (Chopard and Droz, 2005, 2.2.5, p 38-42), gas2.m;
- (second lecture) CA background: definition, neighborhood types, boundary conditions (Chopard and Droz, 2005, 1.3.1-1.3.3, p 12-18, here);
- Systematics: 1d CA, Wolfram rules (Matlab code: ca_1D.m; tables of all rules here and here), (ir)reversibility, Totalistic CA, four universal classes (2.1.1-2.1.2, p 21-26); class 3 and sea shells wikipedia; (see also paper by Wolfram (1984) here).
- Fractals, and fractal dimension (Strogatz, 1994, 11.0-11.4, p 398-410, here).
- More 2d forest fires in a static forest: critical (threshold) tree density, percolation and fractal clusters, universal scaling behavior near percolation threshold (Chopard and Droz, 2005, p 31-33), forest_fire_percolation.m;
- Time permitting: stochastic CA, the diffusion rule and macroscopic limit leading to the 1d diffusion equation (3.1.1 p 67-68; 3.1.3, 71-74).
- Time permitting: other examples: land slides in a sand pile, scale-invariance, $1/f$, universal exponents and behavior as near a phase transition without having to vary a parameter (hence *self-organized* criticality) (Bak et all sandpile model from wikipedia), sandpile_simple.m; rules for sand flow in an hour glass (Chopard and Droz, 2005, 2.2.6 p 42-46).
- Resources: introduction to automata in Matlab.
- *Examples of projects:* snow bumps on ski slopes; sand piles; hour glass;

L20: Third group project presentations

L22-25: Final individual project assignments and progress reports.

L21: THE INTERNET AND GOOGLE'S PAGERANK: [**toolbox concepts:** stochastic matrices, eigenvalues, power method for calculating eigenvalues/ vectors, networks]

- Motivation: building a $100 Billion company based on a simple model; Google vs BMW: this and this.

  - Modeling the Internet via a random walker and the PageRank algorithm from p 1-7 here; the theoretical background, proving that there is a PageRank and that it is unique is the Perron-Frobenius Theorem given in section 6.1 of the file. See also Wikipedia for the theorem and for stochastic matrices; the power method is explained in (Burden and Faires, 2004, 9.2 p 557-558) and in wikipedia.

- During reading period, May 9 and 10: final presentations of individual projects, 15 minutes per presentation.

4 further reading

Norris (1998). Also, relevant material that won't be covered in class: Dixit and Pindyck (1994), Kamien and Schwartz (1991), Ligget (1985).

References

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Chopard, B. and Droz, M. (paperback, 2005). *Cellular automata modeling of physical systems*. Cambridge University Press.

Clark, C. (1976). *Mathematical Bioeconomics*. John Wiley and Sons, New York.

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Fudenberg, D. and Levine, D. K. (1998). *The Theory of Learning in Games*. MIT Press, Cambridge MA.

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Kamien, M. and Schwartz, N. (1991). *Dynamic Optimization*. Elsevier, New York, NY.

Karlin, S. and Taylor, H. (1975). *A First Course in Stochastic Processes, second edition,*. Academic Press, New York.

Ligget, T. (1985). *Interacting Particle Systems*. Springer. (Too advanced for this class, but gives a sense of the many examples of interacting particle systems).

Mesterton-Gibbons, M. (1988). *A concrete approach to mathematical modelling*. Addison-Wesley.

Norris, J. R. (1998). *Markov Chains*. Cambridge University Press, Cambridge UK.

Nowak, M. (2006). *Evolutionary Dynamics*. Harvard University Press, Cambridge MA.

Renshaw, E. (1991). *Modelling biological populations in space and time*. Cambridge University Press.

Strogatz, S. (1994). *Nonlinear dynamics and chaos*. Westview Press.

Whitham, G. B. (1974). *Linear and Nonlinear Waves*. John Wiley and Sons.

Wolfram, S. (1984). Cellular automata as models of complexity. *Nature*, 311:419–424.