Applied Mathematics 105b: Ordinary and Partial Differential Equations

Course web page: http://my.harvard.edu/icb/icb.do?keyword=k72455 *(Spring 2011)*

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Feel free to write call or visit us with any questions.

1 Administrative

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Day & time: MWF, 11-12

Location: Jefferson 250;

Sections: see course web page for times and places.

1st meeting: Monday, Jan 24.

Office hours: Eli: Tuesday 2-3, 24 Oxford, museum building, 4th floor, room 456; Tobias: Thurs 2-3, 29 Oxford Street, 3rd floor, rm 304 / 306; TFs: see course web page.

Textbooks: Page or section numbers from the relevant textbook for any given lecture are given in the detailed syllabus below.

- **Gr** Greenberg, *Advanced Engineering Mathematics*, 2nd edition: our main textbook. ***Any course material not from this textbook will be posted to the downloads directory of the course web page as notes***. the course
- **Kr** Erwin Kreyszig, *Advanced Engineering Mathematics*: very similar contents to Greenberg, somewhat more concise. Will be used occasionally.
- **St** Strogatz, *Nonlinear dynamics and chaos*: will be used for just a few lectures about linear and nonlinear dynamics.
- **Hi** Francis B. Hildebrand Advanced Calculus for Applications (2nd Edition).

Supplementary materials: Additional materials from several additional textbooks and other sources, including Matlab programs used in class, may be found here. Follow links below for the specific source material for each lecture. In order to access these materials from outside the Harvard campus, you'll need to use the VPN software which can be downloaded from the FAS software download site.

Prerequisites: Applied Mathematics 21a and 21b, or Mathematics 21a and 21b.

Sections: Regular times for sections will be scheduled at the beginning of the semester. Each TF will hold a weekly section and have weekly office hours. During the sections, the TFs will discuss and expand on the lecture material and solve additional problems. Although these sections are not mandatory, you are strongly encouraged to attend. Occasionally the TFs will explain material covered in the reading material but not in lectures.

Computer Skills: No programming skills are assumed for this class. Some of the demonstrations, sections and homework assignment will be Matlab-based, and as part of the course, students will therefore gain some experience with this package. We will have introductory Matlab sessions during the first week of the course,

- Tue. Jan 25, 6-8 pm
- Thurs. Jan 27, 6-8 pm

please consult the course web page for details and locations. Students are asked to download and install Matlab on their computers from the FAS software download site.

Homework: Homework will be assigned every Wednesday, and will be due the following Wednesday in class unless otherwise noted. The homework assignments are meant to help you better understand the lecture material and introduce you to come important extensions. It is essential that you actively engage in problem solving using the assigned HW and other problems from the course textbook. Continuously practicing the lecture material on a weekly basis via such problem solving is the only way to become comfortable with the subjects covered in the course.

Midterms, grading: Homework: 30%; three midterms, *tentatively* scheduled to

1. Wednesday 3/2; 7-8:30pm; Jefferson 250
2. Wednesday 3/30; 7-8:30pm; Jefferson 250
3. Wednesday 4/27; 7-8:30pm; Jefferson 250

(all in the evening): 30% together; final: 40%;

Readings: Occasionally we will assign reading material from the textbook or other sources. This material will complement the lectures and is therefore an important part of the course. If not from our main textbook, it can be found under the course downloads web page.

This document: http://www.seas.harvard.edu/climate/eli/Courses/APM105b/2011spring/detailed-syllabus-apm105b.pdf, also available from within campus or via VPN.

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2 Outline

Ordinary differential equations: power series solutions; special functions; eigenfunction expansions. Elementary partial differential equations: separation of variables and series solutions; Introduction to dynamical systems, nonlinear dynamics and chaos. Introduction to numerical methods for solving ordinary and partial differential equations.
Note: Applied Mathematics 105a and 105b are independent courses, and may be taken in any order.

3 Syllabus

Follow links to see the source material and Matlab demo programs used for each lecture. Note that if the source material is not Greenberg, it will be posted as notes or in other format under the appropriate section of the course downloads web page.

1. INTRODUCTION, OVERVIEW. here.
   We'll discuss some logistics, the course requirements, textbooks, overview of the course, what to expect and what not to expect.

2. First order ordinary differential equations. downloads.
Starting simple, this is a subject you may be partially familiar with, but do not fear, unfamiliar material will be showing up very soon...

   (a) Motivation: applications of first order ODEs, and some related images.

      i. Radioactive decay (**Gr**§2.3.2, p 39-end of example 3, p 41).
      ii. Population dynamics, logistic growth (**Gr**§2.3.3, p 41-42).
      iii. Mixing in a tank (**Gr**§2.3.4, p 42-43).
      iv. Earth's energy balance (only derivation of equation) notes.
      v. In sections: Compound interest (**Gr**, exercise 16, p 46); perhaps also circuits with resistor and inductor (**Gr**§2.3.1 starting p 34).

   (b) Introductory remarks (**Gr**§1.1), definitions: ordinary differential equations (ODEs), order of ODEs, system of ODEs, partial differential equations (PDEs), a 'solution', linear vs nonlinear (**Gr**§1.2); just mention at this stage: initial value problems vs boundary value problems (not relevant to 1st order ODEs).

   (c) Linear first order ODEs: $y' + p(x)y = q(x)$.
   Homogeneous case ($q(x) = 0$, **Gr**§2.2.1 to and including example 1).
   Nonhomogeneous case: integrating factor method (notes with example, **Gr**§2.2.2 to example 3). Variation of parameter: useful because it will also be relevant to 2nd order equations (notes with example, **Gr**§2.2.4 to eqn 38).

   (d) Geometric/ dynamical systems approach to nonlinear 1st order ODEs (from Strogatz, see here and notes with examples).

      i. Graphical approach (given $\dot{x} = f(x)$, plot $f(x)$ as function of x and find direction of flow and steady states); fixed points; stability: graphic approach (**St**§2.1, pp 16-18, including Figure 2.2.1)
      ii. Linearized stability analysis (above notes or **St**§2.4, p 24-25, to the end of Example 2.4.1);

   (e) Nonlinear 1st order ODEs.

      i. Equations of separable form (**Gr**§2.4.1 and logistic population model from example 6 in **Gr**§2.4.3);
      ii. Exact differential equations and integrating factors (**Gr**§2.5.1-2.5.2);
      iii. Bernoulli, Riccati, d'Alembert-Lagrange, Clairaut: in sections, time permitting (**Gr**§2.2 exercises 9, 11, 13, 14);

   (f) Existence and uniqueness (**Gr**§2.4.2: theorem 2.4.1 on p 49 and example 4 on pp 51-52).

   (g) Higher order nonlinear ODEs that are reducible by a change of variables to first order equations (second order equations lacking one variable, **Hi**§1.12.5, p 36-37, including examples 9, 10).

i. Only y'', y', x appear in equation, but not y; transform to $p(x) \equiv y'(x)$.

ii. Autonomous equations (x doesn't appear in the equation explicitly) vs non-autonomous (does appear): reduce order by treating $p = y'$ as a function of y: $dp(y)/dx = (dp/dy)(dy/dx) = p(dp/dy)$.

(h) Matlab: symbolic integration and differentiation using Matlab, obtaining an analytic solution to an ODE, and using ode45 for solving ODEs (`symbolic_diff_and_int.m`, `using_dsolve.m`, and `logistic_using_ode45.m` in downloads directory).

(i) Summary.

3. SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS. downloads.
Now we are getting a bit more serious. After a brief motivation and a few examples of second order ODEs, we'll look into the solution of some simpler second order equations, leaving more complex cases to the next section. It is worth noting already now that second (and higher) order ODE's come in two fundamental flavors. If the independent variable is time-like, then we specify "initial conditions" at some initial time, and call the equation an "initial-value problem" (IVP). If, on the other hand, the independent variable is space-like, then we specify "boundary conditions" at particular end points in space, and we call the equation a "boundary-value problem" (BVP). These two types of problem typically have very different solution properties and methods of solution, and we will first consider initial-value problems, which are a natural extension of what you learned about 1st-order problems. We'll consider both initial and boundary value problems several times during the course.

(a) Motivation and examples: The good old $F = ma$, as well as diffusion problems, all lead to 2nd order ODEs. Examples (see also hand-written notes used in class):

i. Linear homogeneous damped pendulum/ spring+mass: $m\ddot{x} + \gamma\dot{x} + \omega^2 x = 0$

ii. Nonlinear homogeneous damped pendulum equation $\ddot{\theta} + \gamma\dot{\theta} + (g/\ell)\sin\theta = 0$

iii. Periodically driven damped nonlinear pendulum and chaos: nonlinear, nonhomogeneous: $\ddot{\theta} + \gamma\dot{\theta} + (g/\ell)\sin\theta = A\cos(\omega t)$.

iv. Steady diffusion of heat in a channel, along one spatial dimension x, with temperature specified at the two ends, also leads to a second order ODE $\frac{d}{dx}\left(k\frac{dT}{dx}\right) + T = 0$. Again a BVP. This could be either diffusion of heat balanced by cooling to the atmosphere, or diffusion supplemented by a combustion reaction! See notes, for now only the case of constant coefficient, related to a channel of constant width and depth.

v. A string between two poles: details later, but for now note that $F = ma$ will lead to a (PDE, from which we will derive a) second order ODE in the spatial dimension, and that "boundary conditions" would need to be specified at both sides of the string (e.g. string does not move there), as opposed to at a single time. That is, a BVP.

(b) Introductory remarks: general linear form $y'' + p(x)y' + q(x)y = r(x)$, linear vs nonlinear, homogeneous vs nonhomogeneous (**Kr**§2.1, pp 45-46); superposition principle (**Kr**§2.1 examples 1-3, theorem 1);

(c) Initial value problems (IVP), basis, general solution (**Kr** pp 47-50 including example 6); briefly contrast with boundary value problems (BVP, **Gr**§3.3.2 pp 88-89), we'll get back to BVPs later.

(d) Reduction of order using a known solution: if $y_1(x)$ is the known solution, try $y_2(x) = u(x)y_1(x)$ and u' then satisfies a 1st order ODE (**Kr** pp 50-52, but use hand written example here instead of the one in **Kr**).

(e) A brief intro to two-dimensional dynamical system perspective: note that this is useful only for autonomous equations;

   i. Writing a second order autonomous ODE as two first order ODEs: for example, in linear constant coefficient case, define $y' \equiv v$, which leads to $v' + pv + qy = r$, together two first order ODEs; note that p, q, r are constants in this case!

   ii. Phase plane analysis is possible and useful if the resulting two first order ODEs are autonomous; discuss possible nonlinear behaviors (stable and unstable fixed points: nodes, saddles, spirals; stable and unstable limit cycles). Linearization around fixed points, linear systems.

   iii. Constant coefficients 2nd order systems, perspective I: geometric approach. First, **St**§5.1 only pages 123-126, to and including figure 5.1.4.

   iv. Motivation for linear constant coefficient 2nd order equations and for the geometric approach in two dimensions: Love affairs: first, start with $\dot{R} = aR + bJ$, $\dot{J} = cR + dJ$, can be written as a single 2nd order equation for R or J. (Substitute $J = [\dot{R} - aR]/b$ into second equation etc, giving $\ddot{R} - \dot{R}(a+d) + R(ad - bc) = 0$.

   v. Next, two simpler examples of love affairs:
      - $\dot{R} = J$, $\dot{J} = -R$,
      - $\dot{R} = -R + J$, $\dot{J} = -R - J$,

      Demonstrate and discuss both using phase space quiver cases 1,2 in love_affairs.m.

   vi. Continue with constant coefficients 2nd order systems: classification of fixed points, stable and unstable spaces (**St**§5.1: example 5.1.2 p 126-128; **St**§5.2 p 129- eqn 6 on p 131).

   vii. Finish with those love affairs, a more complicated example that requires the above machinery: identically cautious lovers: **St**§5.3 pp 138-140, here using cases 3 and 4 in love_affairs.m.

(f) Constant coefficients 2nd order systems, perspective II: briefly: guessing an exponential form, characteristic equation, three cases: two distinct real roots, real double root, complex roots (**Kr** pp 53-57, do only the double real root in some detail because everything else has been covered in the first perspective above; show summary Table on p 57).

(g) In sections, example: damped harmonic oscillator (**Kr** pp 63-66; and example 2, p 77).

(h) (Time permitting) Differential operators, linearity, factorization (**Kr**§2.3)

(i) Euler-Cauchy (equi-dimensional) equations (**Kr**§2.5 pp 69-71); note also that one can transform the independent variable from x to t using $x = e^t$ and the equation is changed to a simple constant coefficient ODE: notes.

(j) Linear independence of solutions, Wronskian (**Kr**§2.6). Note that for vanishing of Wronskian at one point to indicate dependence of two functions, they need to be the solutions of a second order ODE with analytic coefficients.

(k) Nonhomogeneous linear second order ODEs:

   i. Particular solution, general solution being the sum of the general solution to the homogeneous equation and a particular solution (**Kr**§2.7 p 78-79, including theorem 2)

   ii. The method of undetermined coefficients: useful only for a restricted (although important) class of linear equations with constant coefficients a, b in $y'' + ay' + by = r(x)$ and with $r(x)$ a power, exponential, sine/ cosine or a sum of these; perhaps too much of a cookbook style approach (**Kr** pp 79-82)

   iii. (in sections) forced oscillator, **Kr** section 2.8.

   iv. The method of variation of parameters: applies more generally, but is a bit more complex (**Kr**§2.10 pp 98-101)

(l) Summary (**Kr** pp 103-104; plus summary of geometrical approach which is useful in autonomous case only: centers, spirals, nodes, saddles) notes.

- [First midterm hour exam]

4. POWER SERIES SOLUTIONS OF SECOND ORDER LINEAR ODEs, FROBENIUS METHOD, SPECIAL FUNCTIONS. downloads.
In the previous section we were able to solve many problems using exponents, sines and cosines. This, unfortunately, cannot always be done. This section presents a more general approach to solving ODEs based on expanding the solution in a series of polynomials x^n. Exponents, sines and cosines can also be expressed as a series of simple powers, of course. Similarly, several other such expansions lead to what we call "special functions" which are in some sense to be understood later equivalent to exponents, sines and cosines, and which have been cataloged and studied in great detail in the 19th century.

   The power series approach and the accompanying special functions are especially useful in solving *boundary value problems*. There are a few typical cases which are of central importance to mathematical physics and occur very frequently in numerous applications. To slightly generalize, we could state that boundary value problems in simple Cartesian geometry lead to sine and cosine solutions. Cylindrical geometry leads to what's known as "Bessel equation" and "Bessel functions". Similarly, spherical geometry leads to "Legendre equation" and "Legendre polynomials". While we happen to be more familiar with sines and cosines, they share many properties with the Bessel and Legendre functions.

The motivation for much of what we will do now arises in the solution of partial differential equations. We will have a flavor of this motivation in this section, but will see much more when we get to discuss PDEs later in the course.

This section proceeds by first introducing the power series method, then it's application to equations with singular coefficients (that is, coefficients that become infinite, typically at an end point of the interval being considered), and finally discusses the application of the power method to the solution of the Bessel and Legendre equations.

(a) Motivation: the bad news: there is more to life than sines, cosines and exponents. Simple problems lead to problems that cannot be solved using these elementary functions. Examples:

   i. Steady diffusion in a channel with a variable width or depth could lead to a complex, non-autonomous equation that cannot be solved in terms of elementary functions. We already saw the derivation in the case of constant coefficients, consider this time diffusion in a channel with variable depth (notes).

   ii. Diffusion on a disk (derivation later), leads to Bessel equation and functions (figure).

   iii. Diffusion on a sphere (derivation later), leads to Legendre polynomials (figure).

   iv. Sometimes no closed-form solution is known, need to still find a solution and one common format is a power series format $\sum_{n=0}^{\infty} a_n x^n$, see examples of power series representations of elementary and special functions.

(b) Introductory remarks and a basic example (**Gr**§4.1 to equation 6)

(c) Power series review.

   i. Basics (**Gr**§4.2.1, equations 1-6),

   ii. Cauchy's convergence (theorem 4.2.1), interval of convergence and radius of convergence (theorem 4.2.2 on p 178), example 2 on p 178.

   iii. Manipulation of power series, including differentiation, integration, addition, multiplication (theorem 4.2.3 on p 179 and equations 11-14).

(d) Power series solution method around ordinary (non singular) points

   i. Analytic functions, infinite differentiability and Taylor series (**Gr** pp 180-182, including in particular Fig. 2).

   ii. The power series solution method (**Gr**§4.2.2, from p 182 until just before example 5, top of p 185; next, do example 5).

   iii. Summary of the general recipe for obtaining a power series solution to $y'' + p(x)y' + q(x)y = 0$ using Taylor expansion of p, q (**Gr** "closure" pp 189-190).

(e) ODEs with singular coefficients

   i. Motivation for ODEs with singular coefficients:

A. Polar and spherical coordinates are inherently singular (see figures of both polar and spherical), because of the converging grid lines and vanishing grid area/ volume as one approaches the center/ pole. It is therefore not surprising that this leads to ODEs with singular coefficients, derivations will be shown below for the cases of steady diffusion on a disk and sphere.

B. (Time permitting) A singular love affair (notes)

ii. Regular singular points and irregular singular points (definition 4.3.1 on p 194)

iii. Method of Frobenius: series solution of ODEs around regular singular points. Writing the equation as $x^2y''+p(x)xy'+q(x)y=0$, the indicial equation is $r^2+(p_0-1)r+q_0$, where $p(x)=p_0+p_1x+\ldots$ and $q(x)=q_0+q_1x+\ldots$ are analytic. There are three cases for the roots of r as outlined in the theorem:

A. Case I: distinct roots not separated by a *real* integer (this includes complex conjugate roots)

B. Case II: a double root

C. Case III: two roots separated by a real integer

(**Kr**§5.4, pp 183-187. For an alternative which is a bit longer, see **Gr**§4.3.2 including example 3 and to the end of theorem 4.3.1, pp 195-202).

iv. In case II (double root, $r_1=r_2=r$), the second term in the solution $x^r\ln x(a_0+a_1x+\ldots)+x^r(A_1x+A_2x^2+\ldots)$ doesn't start with A_0. Note that we can add y_1 to this solution, and then it would have such a constant term times x^r, so this form is just a convention. In case 3 the two roots are different, so one cannot just add or subtract a constant term times x^{r_1} to the solution proportional to x^{r_2}. As a result, the solution does have an A_0 term.

v. Want to know why an integer difference between the roots is so problematic? Short answer is that in such a case the solution for the smaller of the two roots for the indicial equation leads to a singularity in the recursion relation for the coefficients a_n. Longer answer is in Hildebrand sections 4.4 and 4.5, here.

vi. We showed that cases II and III lead to a log correction using an example and reduction of order. But why is the log correction appearing more generally? Short answer is that in these cases if we try a solution of the form $y(x,s)=x^s\sum_{n=0}^{\infty}a_nx^n$, it can be shown that $\frac{\partial}{\partial s}y(x,s)$ is also a solution, and this derivative brings out the log correction because $\frac{\partial}{\partial s}x^s=(\ln x)x^s$. Details in Hildebrand again, section 4.5.

(f) Using Matlab to find analytic solutions to ODEs (using_dsolve.m)

(g) Legendre's equation $(1-x^2)y''-2xy'+\nu(\nu+1)y=0$ or equivalently $\frac{d}{dx}[(1-x^2)\frac{dy}{dx}]+\nu(\nu+1)y=0$, for $-1\leq x\leq 1$, and Legendre's polynomials $P_n(x)$.

i. Motivation for Legendre polynomials: steady diffusion on a sphere and atmospheric temperature: notes.

ii. Regular singular points at $x=\pm1$ (the poles of the sphere!), series solution about non singular $x=0$ and the case of $\nu=n=$integer leading to Legendre polynomials (**Gr**§4.4.1).

iii. (Time permitting) Orthogonality of $P_n(x)$ (**Gr**§4.4.2).

iv. (Time permitting) Recursion (**Gr** p 215 equation 16), relation involving derivatives (equation 17), and value of integral of $\int_{-1}^{1}[p_n(x)]^2$ (equation 18).

(h) Gamma function (**Gr**§4.5.2, pp 223-225): integral definition for $x > 0$: $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$, and using the recursion relation derived from the integral by integration by parts to define it for negative x: $\Gamma(x) = (x-1)\Gamma(x-1)$.

(i) Bessel equation $x^2y'' + xy' + (x^2 - \nu^2)y = 0$ and functions

i. Motivation for Bessel function: Steady diffusion on a disk, see notes. Why is the equation singular? See again figure of cylindrical coordinates, with converging grid lines and vanishing grid area/ volume as one approaches the center).

ii. Motivation: Bessel functions from a vertically hanging chain: in class demo, animation, notes, and this paper.

iii. Write the equation as $y'' + \frac{1}{x}y' + \frac{(x^2-\nu^2)}{x^2}y = 0$ and consider first the case of large x where the Bessel equation becomes simply $y'' + y = 0$, so that we expect oscillatory behavior for large x (**Gr** p 241, eqn 65 and the following paragraph).

iv. $\nu \neq$integer: solving Bessel's equation using the method of Frobenius (**Gr**§4.6.1, pp 230-232). Solutions to indicial equation are $r = \pm\nu$, and they lead to two independent solutions (need to extend definition of Gamma functions to negative arguments using recursion relation).

v. (very briefly) $\nu =$integer: finding a second linearly independent solution to Bessel's equation, $Y_\nu(x)$, using the Frobenius method, rewriting the second solution to make sure the asymptotic form as $x \to \infty$ is nice and symmetric with that of $J_\nu(x)$ (**Gr**§4.6.2-4.6.3 pp 233-236).

vi. (Time permitting) General properties of the solutions to Bessel's equation: graphical depiction of the solution (plot_Bessel.m); the case of $\nu = 1/2, 3/2, 5/2, \ldots$ where Bessel functions can be expressed in terms of elementary functions (**Kr** p 194-5; **Gr** exercise 4.6.5 p 242)

vii. (Time permitting) Useful properties of Bessel functions: recursion relations between derivatives and different orders (**Kr** p 196; **Gr** exercise 4.6.4 p 242).

viii. (Time permitting) Modified Bessel equation (**Gr**§4.6.5 pp 236-238)

ix. Bessel-like equations. Any equation of the form $\frac{d}{dx}(x^a\frac{dy}{dx}) + bx^cy = 0$ can be transformed to a Bessel equation (**Gr**§4.6.6 example 1, p 238 and then equations 46-50 p 239).

x. (Time permitting) Airy equation $y'' + xy = 0$ (**Gr** p 192, problem 9).

(j) Summary of power series, Frobenius method and special functions (notes).

5. EIGENVALUE (STURM-LIOUVILLE) PROBLEMS AND INTRODUCTION TO FOURIER SERIES. downloads.
"Eigenvalue problems" are basically "boundary value problems" as opposed to the "initial value problems" studies in most of what we did so far. As you will see, this leads to

different physical and mathematical properties of the solutions. When we get to PDEs later in the course, we'll see how eigenvalue problems are helpful in solving initial value problems in PDEs. We will now make use of the Bessel and Legendre functions derived in the previous section using the power series approach.

As we proceed, keep in mind the distinction between boundary value and initial value problems. Notice also the fact already stated above that the geometry of the problem determines which special functions naturally arise in its solution: Cartesian geometry leads to sines and cosines, cylindrical geometry (diffusion on a disk) leads to Bessel functions, and spherical geometry (diffusion on a sphere, say in the earth's atmosphere) leads to Legendre polynomials.

Confusing? Let's just get started and you'll see that it's not that bad.

(a) Introduction

i. Motivation: plucked string and the wave equation: First, Matlab program with numerical simulation of a string: plucked_string_waves_numerical_normal_modes.m, run with `m=1,2,5,10`. Next, notes: write wave equation for the string with no derivation for now, assume periodic in time, and derive a second order ODE and the boundary value problem from the PDE. Proceed using 2nd order boundary value ODEs leading to eigenvalue problems for the frequency (**Gr**§17.7, p 888, example 1); note that *higher frequencies correspond to eigenmodes with more zero crossings*, consistent with the Matlab example. Remark on arbitrary initial conditions and how they would be obtained as a sum of the eigenmodes. Class demo: the string/ spring competition...

(b) Fourier series

i. Even, odd and periodic functions: definitions and some elementary consequences, decomposition of an arbitrary function into even and odd components. (**Gr**§17.2, including examples 1,2).

ii. Fourier series expansion of a periodic function(**Gr**§17.3.1, p 850-851): Stating the Euler formulas for the Fourier coefficients (Eqn 5a,b,c,d), Fourier convergence theorem (theorem 17.3.1); Definition of uniform convergence: p 874 in **Gr**§17.5 including example 1; Gibb's phenomenon is an example for a non uniform convergence.

iii. Example: Fourier series of $x^3 - x$. Fourier_convergence.m

iv. Fourier expansion of a square wave and Gibbs phenomena for functions with a jump discontinuity (**Gr**§17.3.1 example 1, p 852-854): constant %9 overshoot, gibbs.m, meaning of convergence (paragraph starting with in view of this overshoot on p 854). Example: representation of the Andes in spectral atmospheric models.

v. Convergence speed: slowness of convergence for square wave ($1/n$, because of discontinuity, top of p 857). More generally: convergence speed as function of the smoothness of the expanded function (p 856-857, from equation 20 to the end of section 17.3.1). Fourier_convergence.m.

vi. Euler formulas: justification (**Gr**§17.3.2, p 857-8, until Eqn 26), including orthogonality of sines and cosines. A graphical interpretation of this orthogonality. Fourier_orthogonality.pdf.

vii. Half range and quarter range expansions **Gr**§17.4, pp 869-870, and Fig 3 on p 871. (This is also needed later for solving the Laplace equation on a section of a disk, although we may skip that example). Note that the choice of expansion is based on symmetry or anti-symmetry at the end-points of the half-range or quarter-range of the sine/cosine functions, noted by “A” and “S” in Fig 3 on p 871. Matlab demo:
Fourier_half_and_quarter_range.m with arguments
(1,2,0,32), (1,2,1,32), (1,2,2,32) corresponding to example 1, with full range, half range cosine and quarter range cosine expansions.

viii. (In sections, time permitting) Fourier series arising from an ODE: periodically forced oscillator (example 3, pp 859 to end of comment 1, p 861)

(c) Sturm-Liouville 2nd order ODE eigen-problems and generalized orthogonal function expansions

i. Definition of S-L problem: $[p(x)y']' + q(x)y + \lambda w(x)y = 0$ on $a < x < b$, with homogeneous b.c. are $\alpha y(a) + \beta y'(a) = 0$, $\gamma y(b) + \delta y'(b) = 0$, all coefficients are continuous, and $p, w > 0$ on $[a,b]$. Note that b.c. are homogeneous ($= 0$)! λ values which allow non trivial solutions are the *eigenvalues* and the corresponding solutions are the *eigenfunctions*. (**Gr**§17.7.1, pp 887-889 including the simple example 1 ($y'' + \lambda y = 0$) to equation 9).

ii. Some theoretical background:

A. Inner product $\langle f, g \rangle$ (**Gr** p 890, eqn 14 and surrounding paragraph).

B. The Sturm-Liouville theorem (**Gr** theorem 17.7.1a-d, p 891; including example 2 all the way to eqn 18); Note that part d deals with the important completeness property of the eigenfunctions of the Sturm-Liouville problem.

C. Proof of the Lagrange identity and using that of parts of the S-L theorem (real eigenvalues, orthogonal eigenfunctions): (**Gr** pp 898-899).

D. Non negative eigenvalues (**Gr** theorem 17.7.2, p 893); [I got started on notes with the proof, see download directory, but didn’t finish.]

E. Parallels with symmetric matrices (notes, based on the following theorems regarding symmetric matrices (**Gr**§11.3, p 555-557): 11.3.1: eigenvalues are real; 11.3.3: eigenvectors corresponding to distinct eigenvalues are orthogonal; 11.3.4: eigenvectors are a complete orthogonal set to n-space).

iii. A more advanced example: Graphical determination of the eigenvalues (example 3, p 893).

iv. Using integrating factor to bring an equation to S-L form: hand-written notes with complete solution of a case with integrating factor, constant coefficient equation with eigenfunctions of the form exponential times sine/ cosine (or see **Gr** example 4 p 895).

v. Self-adjoint nature of SL operator explains parallels with eigenvectors/ eigenvalues of a symmetric matrix (**Gr**§17.7.2 pp 897-901, end of example 6).

(d) Further motivation: briefly introduce separation of variables of wave equation and show how it leads to a S-L problem. Mention briefly that i.c. need to be expanded in terms of eigenfunctions, hence the importance of completeness (notes).

(e) Additional variants of S-L problems: (1) periodic S-L problems, (2) singular S-L problems ($p=0$ and possibly $w=0$ at one or two end points), an example of which involves the Bessel equation. (3) Infinite domain S-L problems.

i. Periodic S-L problems (**Gr**§17.8, eqn 2, p 906). Note that the general form of the boundary conditions involves p and is more complex then stated in **Gr**, see HW problem on "natural b.c." For periodic S-L problems.

ii. The eigenvalue problem for the Bessel equation (**Gr**§17.8, example 2, pp 908-910).

A. Bring Bessel equation to standard S-L form (**Gr** eqns 46-50 p 238-239).

B. At singular end points, boundedness requirement replaces the homogeneous boundary conditions. Greenberg does not explain this, but it is not difficult to see boundedness is sufficient to make the operator self-adjoint if $p=0$ at the end points of the domain. It is therefore not necessary (and it turns out not possible in some cases) to require the eigenfunctions to vanish at these end points.

C. Orthogonality of Bessel functions corresponding to different eigenvalues, note the weight function $w(x)=x$ in the orthogonality condition: $\int_0^L J_0(z_n x/L) x J_0(z_m x/L)dx = \frac{2\delta_{mn}}{L^2[J_1(z_n)]^2}$ (for the $n=m$ case, see exercise 7 for **Gr**§4.6, p 243).

D. Expansion of a general function in terms of Bessel functions (example 2, p 908-910).

iii. Eigenproblem of Legendre equation on a sphere. Example 3, p 910-911, and remember that we derived the Legendre equation for the problem of steady diffusion in latitude on a sphere, with $x=\sin(\text{latitude})$. Example: heat diffusion on a spherical earth, forced by solar radiation, cooled by long-wave radiation. Assume "zonally-symmetric": no dependence on longitude θ, and also domain limited to the surface itself ("shallow atmosphere" approximation), so no dependence on radius. (notes and Matlab file)

iv. Eigenproblem in an infinite domain: equatorial ocean waves or the Schrodinger equation for a 1d quantum mechanics harmonic oscillator, both lead to the equation $\left[-\frac{h^2}{2m}\frac{d^2}{dx^2}+\frac{K}{2}x^2-E\right]\phi(x)$ with E being the energy eigenvalue, K the spring constant, boundedness requirement at $x \to \pm\infty$ leads to discrete eigenvalues and the solution is Hermite polynomials times a Gaussian $H_n(x)e^{-x^2/2}$).

(f) Review of S-L problems, notes.

6. NUMERICAL METHODS FOR INITIAL VALUE ODE PROBLEMS, WITH A VERY BRIEF INTO NUMERICS OF BOUNDARY VALUE PROBLEMS. downloads.
We are fortunate to have computers and highly user friendly software ("really?" you think to yourself, well friends, it's all relative...) to help us with figuring out what the solution of a given set of equations looks like. While there are courses dedicated to this subject (APM111 is one), we will cover some basics here to give you at least some idea of the rich world of numerical solutions of differential equations.

   (a) Advantages and disadvantages of numerical vs analytic solutions, blurring boundaries between the two for complex analytic solutions, numerical sensitivity vs analytic expression for dependence on parameters.

   (b) Euler method, estimates of error and accumulated error (**Gr**§6.2 pp 293-296 to eqn 11; Taylor Formula is derived on p 631, eqns 7a-10); definition of order p method: accumulated truncation error is $E_n = O(h^p)$; choosing a sufficiently small h (p 297); Matlab demo: euler_error.m.

   (c) Leap-frog (mid-point) method (**Gr**§6.3.1 p 299-301); improved Euler method, also known as Runge-Kutta method of second order (**Gr**§6.3.2, p 302-304).

   (d) Introduction to numerical (in)stability: instability due to existence of a second solution (**Gr**§6.5 example 1, p 323), numerical_instability_2nd_solution.m; instability due to the scheme itself with leap-frog (mid-point) as an example (example 2, p 324), numerical_instability_leap_frog.m.

7. VECTOR CALCULUS: A VERY BRIEF REMINDER. downloads.
While we assume you have seen the gradient, divergence and Laplacian operators of vector calculus in previous courses (e.g., APM21a,b), we will now briefly remind you what they are and especially attempt to provide a physical intuition for each of these. We will also discuss the derivation of these in cylindrical and spherical coordinates. This is all in preparation for the discussion of partial differential equations next.

   (a) Motivation: bees do it? (on a mac, this) and then, yes, bees are using vector calculus.

   (b) Introduction and review: A quick reminder

      i. Two and three dimensional scalar fields (temperature) and vector fields (wind, heat flux).

      ii. Dot product (**Gr**§14.2 eqn 1, pp 683)

      iii. Cartesian coordinates (**Gr**§14.3 pp 687-690, but only eqns 1, 5, 9, 10, 11d)

   (c) Vector calculus: div: $\nabla \cdot \vec{a}$, grad: $\nabla\phi$, Laplacian: $\nabla^2 a$, $\nabla^2\vec{a}$.

i. Divergence: definition using a general closed-surface integral over a vector field at the limit of the surface becoming infinitesimal. Derivation in Cartesian for a cube-like surface, and the differential operator. (**Gr**§16.3, pp 761-765, not including examples).

ii. Gradient: definition. Overview of input and output of div, grad (& curl); directional derivative Geometric meaning of the gradient: Its direction is perpendicular to surfaces of constant value of the scalar field, and magnitude is equal to the directional derivative in this perpendicular direction. (**Gr**§16.4, pp 766-769 until but not including example 3).

iii. Providing physical intuition for divergence: mass conservation for an incompressible fluid: $\nabla \cdot \mathbf{u} = 0$ (**Gr** p 797-799, example 2, although that example derives the full compressible continuity equation, so skip all that and discuss an appropriately simpler version).

iv. Laplacian (**Gr**§16.6, pp 779-780, equations at bottom of page 779 and then eqns 9,10).

v. Providing physical intuition for grad, div and Laplacian: temperature field $T(x,y,x)$, diffusive heat flux vector field $k\nabla T = k(T_x, T_y, T_z)$, diffusive local heating rate given by Laplacian, $div(grad(T)) = \nabla^2 T = T_{xx} + T_{yy} + T_{zz}$ (derive this). vector_calculus_preliminaries.m;

(d) Vector calculus in orthogonal curvilinear coordinates

i. Polar coordinates and (briefly) the derivatives of unit vectors with respect to the different coordinates, e.g. $\partial \hat{\mathbf{e}}_r / \partial \theta$, used later for deriving the divergence and Laplacian, (**Gr**§14.6.1, pp 700-704, but only eqns 4a,5, 10, 11, Fig 2, 15, 16, 18, Fig 3).

ii. Cylindrical coordinates (first paragraph of **Gr**§14.6.2, p 704 and Fig 5 there). examples where cylindrical coordinates are useful (first half of p 705).

iii. Spherical coordinates (**Gr**§14.6.3, p 705-706, to end of eqns 28).

(e) ∇ in non Cartesian coordinates

i. Cylindrical coordinates: use the **Gr** derivation of ∇ (**Gr**§16.7.1 p 783); and then use my notes for deriving the divergence. Then just write down the Laplacian (**Gr** p 785), explain the need to use $\partial \hat{\mathbf{e}}_r / \partial \theta$ and friends, as mentioned at the bottom first page of my notes.

ii. Spherical coordinates: basically just write down the results for the gradient, divergence and Laplacian (**Gr**§16.7.2 pp 786-787, eqns 27-32).

- [Second midterm hour exam]

8. INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATIONS. Including an introduction to both analytic and (briefly) numerical methods. downloads.
Many if not most of the equations encountered in applications are partial differential equations. This applies to mathematical physics, the Geosciences, fluid dynamics, many

biological applications, economy and more. Luckily for us, the solution of these equations, whether time dependent or not, is based on the theory we have developed so far, especially that involved in boundary value problems, eigenfunction expansion using sines, cosines, Bessel and Legendre functions. Our hard work on these will finally pay off (and the reward will be, of course, even more work) when we see how these previous concepts naturally arise and become most helpful now.

(a) Motivation: fluid dynamics, quantum mechanics, electrodynamics, diffusion, waves, are all governed by PDEs. Specific examples are given below for the different kinds of PDEs.

(b) Introduction, basics

  i. Basic examples, general solution involving arbitrary functions rather than the arbitrary constants in ODE case (**Kr**§12.1, everything to the end of example 3, pp 535-537).

  ii. Classification of linear second order PDEs in two independent variables (**Gr**§18.2.2, p 946-948 to end of example 3): $Au_{xx}+2Bu_{xy}+Cu_{yy}+Du_x+Eu_y+Fu=f$, (coefficients may be functions of independent variables). Consider the prototypical PDEs and the related physical problems,

    A. $B^2-AC=0$: parabolic, example: diffusion equation: $u_t=\alpha^2 u_{xx}$,

    B. $B^2-AC>0$: hyperbolic, example: wave equation: $c^2u_{xx}=u_{tt}$,

    C. $B^2-AC<0$: elliptic, example: Laplace equation: $u_{xx}+u_{yy}=0$.

  iii. (Time permitting) what's behind the classification: characteristics of linear second order equations (Hildebrand section 8.8, pp 408-414).

  iv. Boundary conditions: Dirichlet (u prescribed), Neumann ($\frac{\partial u}{\partial n}$ prescribed), and mixed (**Gr** p 951-952).

(c) Diffusion equation $u_t=\kappa u_{xx}$

  i. Motivation via diffusion of AIDS, computer virus, Walmart, heat in computer chips, all here.

  ii. Derivation of the one dimensional time dependent diffusion equation (using the heat budget of a one dimensional or two dimensional infinitesimal element: see notes; or (not so good in this case) **Gr**§18.2.3, p 948-949)

  iii. Class demo of diffusion: smelling the Black Musk deer.

  iv. Scaling argument for diffusion of perfume, $C(x,t)$, in air: the diffusion time scale. $\frac{\Delta C}{\tau}\sim\kappa\frac{\Delta C}{L^2}$, so that $\tau{\sim}L^2/\kappa$ and $L\sim\sqrt{\kappa\tau}$. for perfume in class: if $L=10m$, using $\kappa=0.2\times10^{-4}m^2/s$, we get $\tau\sim5\times10^6 sec$ which is roughly a few weeks long...

  v. Derivation of the 2d diffusion equation: notes.

  vi. Solving the time dependent 1d diffusion problem using separation of variables

    A. **Gr**§18.3.1, example 1: from eqn 1a,b,c to 28, pp 954-958. Comment 1, p 959 interpreting the first term in eqn 26 as the steady state solution.

B. Demonstrate behavior of time dependent solution using diffusion_1d_SL.m

C. The physics of Cappuccino, in-class demo: Comment 2, p 959, discussing the scale-selective nature of diffusion. Demo of food colors in water on overhead. Use above Matlab script (icase of 4,5) to discuss scale selective character of diffusion. In-class demonstration of stirring and mixing.

D. Connection to S-L: Note the need to first solve for and subtract the steady solution in order for the resulting S-L problem to be homogeneous as required.

E. Need to apply the b.c. before the initial conditions, because b.c. lead to S-L problem, which can then be used to expand arbitrary i.c. (**Gr** p 960, comment 4).

F. Note that the final solution is not a function of time times a function of space, although each term in the expansion is. The form $X(x)T(t)$ would be too restrictive as the initial spatial structure would not be able to change expect overall time dependent scale (comment 6, p 960).

G. Verification of the solution (**Gr**§18.3.2, only eqns 49a,b,c and the following single sentence, p 964).

H. Example 4, p 967: in sections.

vii. Heat equation on a disk, in cylindrical coordinates (**Gr**§18.3.3, example 5, pp 968-971, including comments 1,2; may need to also use example 1 in section **Gr**§4.6.6 p 238 to show the transformation to the standard Bessel form, and perhaps example 2 in **Gr**§17.8 p 908 for the needed S-L expansion using Bessel functions). Demonstrate behavior of time dependent solution using diffusion_2d_disk_SL.m.

viii. Uniqueness of solutions to diffusion equation (**Gr** exercise 25, p 979). This is an example of using an integral constraint when studying a PDE, which is often a useful approach. We'll see this again below when we deal with the steady-state diffusion equation in 2d with Neumann b.c., leading to Laplace equation.

ix. When separation of variables cannot be used: an example of a non separable equation: $\nabla^4 u = u_{xxxx} + 2u_{xxyy} + u_{yyyy} = 0$, and where the boundary conditions are not separable (**Gr** p 972).

x. Numerical solution of the diffusion equation in 1d and 2d

A. Euler forward in time, center difference in space, including a (possibly moving) source. Show both 1d and 2d schemes, and demonstrate using diffusion_1d_numerical.m, diffusion_2d_numerical.m.

(d) Wave equation $u_{tt} = c^2 u_{xx}$

i. Motivation: tidal bore wave, beach waves, surfing, sounds waves forcing drum vibrations, vibrating membrane animations from Wikipedia, all here.

ii. Derivation of wave equation for a string $u_{tt} = c^2 u_{xx}$ (**Gr**§19.1, pp 1017-1019), ignore the forcing term included there.

iii. State without derivation the two spatial dimensions version: $u_{tt} = c^2\nabla^2 u$.

iv. More motivation: stadium waves, and derivation of equation.

v. 1d waves

A. Separation of variables: full solution of the plucked string problem using Fourier series. Calculate coefficients in Fourier series based on boundary and initial conditions. show specific example based on plucked string (**Gr**§19.2.1, p 1023-1026, to Eqn 21).

B. Traveling wave interpretation (**Gr**§19.2.2 to Eqn 30 and including Figs 4 and 5, pp 1027-1029).

C. Numerical solution of wave equation using center differences in space and time, compare series and numerical solutions (plucked_string_waves_numerical.m, plucked_string_waves_analytic.m).

D. (Time permitting) wave equation for shallow water (misc downloads and notes)

E. D'Alembert's solution to the 1d wave equation, general solution of the initial value problem (**Gr**§19.4.1 eqns 1-7 and then example 1, pp 1043-1048).

vi. A 2d wave equation:

A. Motivation: sounds wave visualization on a membrane. Animation of modes on a circular membrane from Wikipedia here.

B. Vibration modes of a rectangular membrane (**Gr**§19.3 p 1035-1039, end of comment 2); Matlab demo for a rectangular membrane (try both `plain_surface`=1 and 0).

C. In class demo: bed sheet with 8 volunteers, and metronome from www.metronomeonline.com/.

(e) Laplace's equation $\nabla^2 u = u_{xx} + u_{yy} = 0$ (**Gr** chapter 20, p 1058)

i. Motivation for Laplace equation: diffusion equation in 2d at steady state with or without a distributed specified heat source (**Gr**§20.1, pp 1058-1059).

ii. Another motivation: potential flow around a cylinder, start with an airplane wing, transition to a cylinder, see notes.

iii. Cartesian coordinates:

A. Separation of variables, choosing the "right" sign for the separation variable, superposition (**Gr**§20.1, example 1, skip comment 1, include comment 2, pp 1060-1063).

B. Choosing the right sign of the separation variable, discussion after comment 3 on p 1062.

C. See also discussion of the case in which there are nonhomogeneous b.c., which requires expansion at more than one side of the domain (last 2 lines of p 1062 to p 1063 just before example 2).

iv. Consistency requirement with Neumann b.c. (derivative specified) for Poisson's equation $\nabla^2 u = f(x,y)$: such a b.c. may on all sides can lead to no solution if not formulated carefully. Physically this just means that the prescribed heat source $f(x,y)$ plus the heat flux into the domain via the prescribed Neumann b.c. must sum to zero for a steady state solution to the diffusion equation to be possible (**Gr** Exercise 17, p 1069). This demonstrates again the usefulness of integral constraints in PDEs.

v. Laplace's equation in cylindrical geometries $(u_{zz}+)\,u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0$ (**Gr**§20.3, p 1070)

   A. Definitions of plane polar coordinates ($\partial/\partial z = 0$) vs cylindrical coordinates (contrast §20.3.1 eqn 1 with §20.3.2, eqn 37, pp 1070 and 1077). We shall refer to both as cylindrical.

   B. Laplace equation with Dirichlet (specified boundary values) on a full disk, using example 2: First, only the beginning of example 1, p 1070-1071 which is needed later for example 2: only equations (1,2a on full disk, 3-9) until the end of the derivation of the general solution. Next, example 2, Dirichlet (specified boundary values) on a full disk, using boundedness and periodicity instead of some of the b.c. (p 1073-1076 to eqn 34).

   C. The maximum principle of the Laplace equation. Also, the average value property. Physical interpretation of the maximum principle in terms of temperature distribution and diffusive heat fluxes (comment 1 on p 1075-1076, and then text from end of comment 2 to end of §20.3.1, pp 1076-1077).

   D. Potential flow around a cylinder (notes).

vi. (Time permitting) Laplace's equation in spherical coordinates and Legendre polynomials (**Gr**§20.3.3, p 1081-1083); Coordinates are (ρ,ϕ,θ), where θ is longitude-like, and ϕ latitude-like, except that $\phi=0$ at the north pole. Laplace equation is

$$\nabla^2 u = \frac{1}{\rho^2}\left[\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial u}{\partial\rho}\right)+\frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial u}{\partial\phi}\right)+\frac{1}{\sin^2\phi}\frac{\partial^2 u}{\partial\theta^2}\right]=0$$

(f) Review of PDEs

9. DYNAMICAL SYSTEMS, NONLINEAR DYNAMICS AND CHAOS. downloads.
With a few exceptions, most of we did so far involved linear equations. (Un?)fortunately, natural systems are more often than not nonlinear. While approximating them as linear allows us to obtain a solution and perform a much more complete analysis, this also prevents us from seeing the full richer behaviors these systems are capable of. A more complete treatment of these subjects is given in other courses (e.g. APM147), yet we feel it is important for you to get at least a flavor of the fascinating phenomena allowed by nonlinearity. Time allowing, we'll learn about bifurcations, synchronization, chaos and

fractals. This subject is certainly one of our favorites. Finally, you'll see that dynamical systems are all about *initial value* problems.

(a) *Bifurcations:* saddle node bifurcation, example of global climate, catastrophe and hysteresis, letter to the president; (**St**§3.1, p 45-47, to end of example 3.1.1; alternatively, p 47 of my notes; 0d energy balance notes, see also animation of bifurcation here).

(b) *Synchronization:* nonlinear phase locking and synchronization, fireflies, Huygens clocks, electric grid, El Nino, Glacial cycles, and best of all: youtube of metronomes on coke cans. (p 75 and 77 of my notes)

(c) *Chaos:*

   i. Motivation, history: (•) Hyperion, moon of Saturn; three body problem competition for King Oscar's birthday; Poincare and the the mistake in the winning entry and discovery of chaos. (•) Lorenz and the mistake in the restarting from printed output and the discovery of dissipative chaos and strange attractors.

   ii. Brief mention of Poincare-Bendixson theorem: need at least three dimensions for chaos (**St**§7.3 p 203-204).

   iii. Lorenz equations, character of the solution via time series and phase space, sensitivity to initial conditions java applet; limited predictability; (my notes, pp 95,99,101; or **St**§9.3 p 317-end of example 9.3.1 p 323)

   iv. Definition of chaos (p 101 in notes or **St** p 323-324)

   v. Logistic map, chaos, orbit diagram, period doubling route to chaos, universal constants, periodic windows; (pp 106, 107, 108, 109, 110, 112 in notes)

   vi. (Time permitting) Poincare sections, maps vs continuous dynamical systems, why maps can display such rich behavior (**St**§8.7 p 278 - to upper paragraph of p 278).

(d) *Fractals:*

   i. Motivation: fractal leaves and clouds, as well as strange attractors.

   ii. Cantor set (**St**§11.2 p 401-402), von Koch curve (Fig 11.3.1 p 405), fractal box dimension (**St**§11.4 p 409-410 including dimensions of cantor set, von Koch curve and example 11.4.2).

   iii. Connection between chaos and fractals (pp 237,239 in my notes)

10. REVIEW
Summary map for solving a 2nd order linear PDE, demonstrating many concepts covered during the course, and notes for the Schrodinger equation example which again demonstrates many useful subjects covered.